Integrable models from the Boundary Quantum Inverse Scattering Method

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Outline

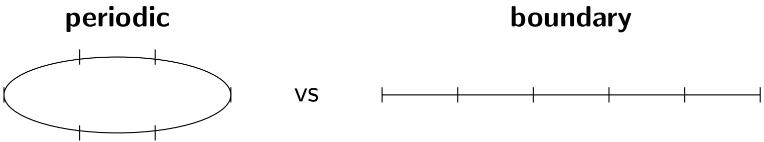
- **Quantum Inverse Scattering Method** for "twisted" periodic boundary conditions.
  - Integrable models from the quasi-classical limit.
- **Boundary Quantum Inverse Scattering Method** for open-boundary conditions.
  - Integrable models from the quasi-classical limit.

Some History

- **Quantum Inverse Scattering Method (QISM)** for periodic boundary conditions [Faddeev, Kulish, Sklyanin, Takhtajan 1979].
- **BCS model** of superconductivity [Bardeen, Cooper and Schrieffer 1957].
  - Solved for a particular case: **Richardson model** [Richardson 1963].
  - **Integrals of motion** for the Richardson model [Cambiaggio et al. 1997].
  - **Eigenvalues** for the Richardson model [Sierra 2000].
  - Generalised to the **trigonometric case** [Amico et al. 2001], [Dukelsky et al. 2001].
- **Reformulated** through QISM [Zhou et al. 2002], [von Delft and Poghossian 2002].
- **Some extensions:** [Ovchinnikov 2003], [Dunning and Links 2004], [Ibañez et al. 2009], [Skrypnyk 2009], [Dukelsky et al. 2010, 2011], [Links and Marquette 2013].

Some History

- **Boundary QISM** for open-boundary conditions, for the case of XXZ spin chain [Sklyanin 1988].



- **Gaudin magnet** with boundary [Hikami 1995].
- **Quasi-classical limit** of the BQISM [Di Lorenzo et al. 2002].
- **Generalized** Gaudin systems [Skrypnyk 2006, 2007, 2010].
- **Trigonometric** Gaudin model [Cirilio António, Manilović and Nagy 2013].

R-matrix

R-**matrix** is an operator $R(u) \in \mathrm{End}(V \otimes V)$ ($V \cong \mathbb{C}^2$, $u \in \mathbb{C}$) satisfying the **Yang-Baxter equation** (YBE) in $\mathrm{End}(V \otimes V \otimes V)$:

$$\boxed{R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v).}$$

I. Trigonometric solution

$$R^{trig}(u) = \frac{1}{\sinh(u+\eta)} \begin{pmatrix} \sinh(u+\eta) & 0 & 0 & 0 \\ 0 & \sinh u & \sinh\eta & 0 \\ 0 & \sinh\eta & \sinh u & 0 \\ 0 & 0 & 0 & \sinh(u+\eta) \end{pmatrix}.$$

II. Rational solution ($\lim_{\xi\to 0} \dfrac{\sinh(\xi\alpha)}{\xi} = \alpha$ from the trigonometric)

$$R^{rat}(u) = \frac{1}{u+\eta}(uI \otimes I + \eta P) = \frac{1}{u+\eta} \begin{pmatrix} u+\eta & 0 & 0 & 0 \\ 0 & u & \eta & 0 \\ 0 & \eta & u & 0 \\ 0 & 0 & 0 & u+\eta \end{pmatrix},$$

P is the permutation operator: $P(u \otimes v) = v \otimes u, \ \forall u, v \in V.$

QISM [Faddeev et al. 1979]

Monodromy matrix (acts in $V_a \otimes V^{\otimes \mathcal{L}}$, $V_a = \mathbb{C}^2$ auxiliary space)

$$T_a(u) \equiv \begin{pmatrix} e^{-\eta\alpha} & 0 \\ 0 & e^{\eta\alpha} \end{pmatrix} R_{a\mathcal{L}}(u-\varepsilon_{\mathcal{L}})...R_{a2}(u-\varepsilon_2)R_{a1}(u-\varepsilon_1) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

satisfies the Yang-Baxter equation in $\mathrm{End}(V_a \otimes V_b \otimes V^{\otimes \mathcal{L}})$:

$$R_{ab}(u-v)T_a(u)T_b(v) = T_b(v)T_a(u)R_{ab}(u-v).$$

Transfer matrix $t(u) \equiv tr_a(T_a(u)) = A(u) + D(u) \in \mathrm{End}(V^{\otimes \mathcal{L}})$,

$$\boxed{[t(u), t(v)] = 0 \quad \forall u, v \in \mathbb{C}.}$$

Consider the expansion of $t(u)$ in powers of u:

$$t(u) = \sum_{j=-\infty}^{\infty} C_j u^j.$$

Then C_j form a set of mutually commuting **integrals of motion**.

Bethe Ansatz [Bethe 1931]

Start with a **reference state**, $\Omega \in V^{\otimes \mathcal{L}}$, such that

$$T(u)\Omega = \begin{pmatrix} a(u) & 0 \\ * & d(u) \end{pmatrix} \Omega.$$

Then (for the **trigonometric** R-matrix)

$$\Phi(v_1, ..., v_N) = C(v_1)...C(v_N)\Omega$$

is an **eigenstate** of $t(u)$ with the **eigenvalue**

$$\Lambda(u, v_1, ..., v_N) = a(u) \prod_{k=1}^{N} \frac{\sinh(u - v_k + \eta)}{\sinh(u - v_k)} + d(u) \prod_{k=1}^{N} \frac{\sinh(u - v_k - \eta)}{\sinh(u - v_k)},$$

if $\Phi \neq 0$ and v's satisfy the **Bethe Ansatz equations**

$$\frac{a(v_k)}{d(v_k)} = \prod_{j \neq k}^{N} \frac{\sinh(v_k - v_j - \eta)}{\sinh(v_k - v_j + \eta)}, \quad k = 1, ..., N.$$

For the **rational** R-matrix: $\sinh(x) \to x$.

Quasi-classical limit

Take the first non-zero term in the expansion as $\eta \to 0$:

$$\lim_{u \to \varepsilon_j} (u - \varepsilon_j) t(u) = \eta^2 \tau_j + o(\eta^2).$$

Then τ_j are the integrals of motion. They can be expressed in terms of the **spin operators**:

$$S^x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; S^y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; S^z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$S^+ = S^x + iS^y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; S^- = S^x - iS^y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

They generate the Lie algebra $\mathfrak{su}(2)$. **Commutation relations:**

$$[S^z, S^\pm] = \pm S^\pm, \; [S^+, S^-] = 2S^z.$$

I. Trigonometric case [Amico et al. 2001], [Dukelsky et al. 2001]

Notation: $v_j = \ln(y_j), \ \varepsilon_l = \ln(z_l)$

- **BAE:**

$$(\alpha + N - 1 - \mathcal{L}/2) + \sum_{l=1}^{\mathcal{L}} \frac{y_k^2}{y_k^2 - y_j^2} = 2\sum_{j\neq k}^{N} \frac{y_k^2}{y_k^2 - y_j^2}.$$

- **Integrals of motion:**

$$\tau_j = 2\sum_{k\neq j}^{\mathcal{L}} \frac{z_j^2 + z_k^2}{z_j^2 - z_k^2} S_j^z S_k^z + \sum_{k\neq j}^{\mathcal{L}} \frac{2z_k z_j}{z_j^2 - z_k^2} \left(S_k^- S_j^+ + S_k^+ S_j^- \right) - 2\alpha S_j^z.$$

- **Eigenvalues:**

$$\lambda_j = \frac{1}{2} \sum_{k\neq j}^{\mathcal{L}} \frac{z_j^2 + z_k^2}{z_j^2 - z_k^2} - \sum_{i=1}^{N} \frac{z_j^2 + y_i^2}{z_j^2 - y_i^2} + \alpha.$$

II. Rational case [Richardson 1963], [Cambiaggio et al. 1997], [Sierra 2000]

- **BAE:**

$$2\alpha + \sum_{l=1}^{\mathcal{L}} \frac{1}{v_k - \varepsilon_l} = \sum_{j \neq k}^{N} \frac{2}{v_k - v_j}.$$

- **Integrals of motion:**

$$\tau_j = \sum_{k \neq j}^{\mathcal{L}} \frac{2S_k^z S_j^z + S_k^- S_j^+ + S_k^+ S_j^-}{\varepsilon_j - \varepsilon_k} - 2\alpha S_j^z.$$

- **Eigenvalues:**

$$\lambda_j = \frac{1}{2} \sum_{k \neq j}^{\mathcal{L}} \frac{1}{\varepsilon_j - \varepsilon_k} - \sum_{i=1}^{N} \frac{1}{\varepsilon_j - v_i} + \alpha.$$

Note: these can be obtained as a limit from the trigonometric.

Reflection equations [Cherednik 1984]

Start with a **rational** R-matrix $R(u) = \dfrac{1}{u+\eta}(uI \otimes I + \eta P)$.

In addition to YBE we want it to satisfy the **reflection equations**:

$$\begin{cases} R_{12}(u-v)K_1^-(u)R_{21}(u+v)K_2^-(v) = K_2^-(v)R_{12}(u+v)K_1^-(u)R_{21}(u-v), \\ R_{12}(v-u)K_1^+(u)\mathcal{R}_{21}(u+v)K_2^+(v) = K_2^+(v)\mathcal{R}_{12}(u+v)K_1^+(u)R_{21}(v-u), \end{cases}$$

where $\mathcal{R}(u) \equiv R(-u-2\eta)$. Easy to check that

$$K^-(u) = \zeta I + 2uS^z = \begin{pmatrix} \zeta + u & 0 \\ 0 & \zeta - u \end{pmatrix},$$

$$K^+(u) = \xi I + 2(u+\eta)S^z = \begin{pmatrix} \xi + u + \eta & 0 \\ 0 & \xi - u - \eta \end{pmatrix}.$$

satisfy these equations. They are called the **reflection matrices**.

BQISM [Sklyanin 1988]

Define the **double monodromy matrix** (acting in $V_a \otimes V^{\otimes \mathcal{L}}$)

$$T_a(u) \equiv R_{a\mathcal{L}}(u-\varepsilon_{\mathcal{L}})\ldots R_{a1}(u-\varepsilon_1)K_a^-(u)\times$$
$$\times R_{a1}^{-1}(-u-\varepsilon_1)\ldots R_{a\mathcal{L}}^{-1}(-u-\varepsilon_{\mathcal{L}}) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}.$$

It satisfies the reflection equation in $V_a \otimes V_b \otimes V^{\otimes \mathcal{L}}$:

$$R_{ab}(u-v)T_a(u)R_{ba}(u+v)T_b(v) = T_b(v)R_{ab}(u+v)T_a(u)R_{ba}(u-v).$$

Define the **transfer matrix**

$$t(u) \equiv tr_a\left(K_a^+(u)T_a(u)\right) = (\xi+u+\eta)A(u) + (\xi-u-\eta)D(u).$$

In this case one can also show that

$$[t(u), t(v)] = 0 \quad \forall u, v \in \mathbb{C}.$$

A New Unified Approach

Define the **double monodromy matrix** (acting in $V_a \otimes V^{\otimes \mathcal{L}}$)

$$T_a(u) \equiv R_{a\mathcal{L}}(u-\varepsilon_{\mathcal{L}}) \ldots R_{a1}(u-\varepsilon_1) K_a^-(u+\rho/2) \times$$
$$\times R_{a1}^{-1}(-u-\varepsilon_1-\rho) \ldots R_{a\mathcal{L}}^{-1}(-u-\varepsilon_{\mathcal{L}}-\rho) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}.$$

It satisfies the reflection equation in $V_a \otimes V_b \otimes V^{\otimes \mathcal{L}}$:

$$R_{ab}(u-v) T_a(u) R_{ba}(u+v+\rho) T_b(v) = T_b(v) R_{ab}(u+v+\rho) T_a(u) R_{ba}(u-v).$$

Note: $\rho \to \infty$ yields the YBE as in periodic case

$$R_{ab}(u-v) T_a(u) T_b(v) = T_b(v) T_a(u) R_{ba}(u-v);$$

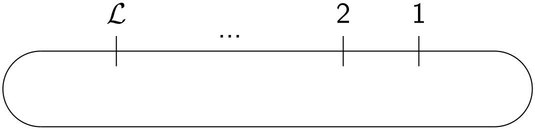
when $\rho \to 0$ we obtain Sklyanin's formulation.

The **transfer matrix:** $t(u) \equiv tr_a \left(K_a^+(u+\rho/2) T_a(u)\right)$.

Periodic vs Boundary

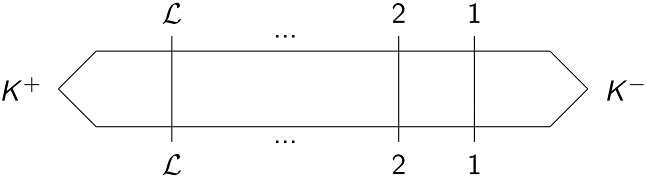
- **Periodic:**

$$t(u) = tr_a \left(R_{a\mathcal{L}}(u-\varepsilon_{\mathcal{L}}) \ldots R_{a2}(u-\varepsilon_2) R_{a1}(u-\varepsilon_1) \right).$$



- **Boundary:**

$$t(u) = tr_a \Big(K_a^+(u) R_{a\mathcal{L}}(u-\varepsilon_{\mathcal{L}}) \ldots R_{a1}(u-\varepsilon_1) K_a^-(u) \times \times R_{a1}^{-1}(-u-\varepsilon_1) \ldots R_{a\mathcal{L}}^{-1}(-u-\varepsilon_{\mathcal{L}}) \Big).$$



Bethe Ansatz

Shift $u \to u - \frac{\eta}{2}$ and introduce $\tilde{a}(u) = (2u+\rho)a(u) - \eta d(u)$.

$$\Phi(v_1, ..., v_N) = C(v_1)...C(v_N)\Omega$$

is an **eigenstate** of $t(u)$ with the **eigenvalue**

$$\Lambda(u, v_1, ..., v_N) = \tilde{a}(u)\frac{\xi + u + \rho/2 + \eta/2}{2u+\rho}\prod_{k=1}^{N}\frac{(u - v_k + \eta)(u + v_k + \rho + \eta)}{(u - v_k)(u + v_k + \rho)} +$$

$$+ d(u)\frac{(2u+\rho+\eta)(\xi - u - \rho/2 + \eta/2)}{2u+\rho}\prod_{k=1}^{N}\frac{(u - v_k - \eta)(u + v_k + \rho - \eta)}{(u - v_k)(u + v_k + \rho)},$$

if $\Phi \neq 0$ and v's satisfy the **Bethe Ansatz equations (BAE)**

$$\frac{\tilde{a}(v_k)}{d(v_k)(2v_k + \rho - \eta)}\frac{\xi + v_k + \rho/2 + \eta/2}{\xi - v_k - \rho/2 + \eta/2} = \prod_{j\neq k}^{N}\frac{(v_k - v_j - \eta)(v_k + v_j + \rho - \eta)}{(v_k - v_j + \eta)(v_k + v_j + \rho + \eta)}.$$

Can **renormalize**, so that it goes to non-boundary case as $\rho \to \infty$.

Quasi-classical limit

Take $\Omega = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{\otimes \mathcal{L}}$ as a reference state and compute $\tilde{a}(u)$ and $d(u)$. Then

$$\frac{\tilde{a}(v_k)}{d(v_k)(2v_k + \rho - \eta)} = \frac{\zeta + v_k + \rho/2 + \eta/2}{\zeta - v_k - \rho/2 + \eta/2} \prod_{l=1}^{\mathcal{L}} \frac{(v_k - \varepsilon_l - \eta/2)(v_k + \varepsilon_l + \rho - \eta/2)}{(v_k - \varepsilon_l + \eta/2)(v_k + \varepsilon_l + \rho + \eta/2)}.$$

If we substitute $\eta = 0$ the BAE will take the following form:

$$\frac{(\zeta + v_k + \rho/2)(\xi + v_k + \rho/2)}{(\zeta - v_k - \rho/2)(\xi - v_k - \rho/2)} = 1.$$

We want to choose $\zeta = \zeta(\eta),\ \xi = \xi(\eta)$, so that this holds as $\eta \to 0$.

Consider **two cases:**

A. $\xi = \eta\alpha, \quad \zeta = \eta\beta.$

B. $\xi = \eta^{-1}\gamma^{-1}, \quad \zeta = \eta^{-1}\delta^{-1}.$

A. $\xi = \eta\alpha,\ \zeta = \eta\beta$: same as from trigonometric R-matrix in periodic case!

Notation: $y_k = v_k + \frac{\rho}{2},\ z_l = \varepsilon_l + \frac{\rho}{2}$.

- **BAE:**

$$-(\alpha+\beta+1) + \sum_{l=1}^{\mathcal{L}} \frac{y_k^2}{y_k^2 - z_l^2} = \sum_{j\neq k}^{N} \frac{2y_k^2}{y_k^2 - y_j^2}.$$

- **Integrals of motion:**

$$\tau_j = 2\sum_{k\neq j}^{\mathcal{L}} \frac{z_j^2 + z_k^2}{z_j^2 - z_k^2} S_j^z S_k^z + \sum_{k\neq j}^{\mathcal{L}} \frac{2z_j z_k}{z_j^2 - z_k^2}\left(S_k^- S_j^+ + S_k^+ S_j^-\right) +$$
$$+ 2\left(N - \mathcal{L}/2 + \alpha + \beta\right) S_j^z.$$

- **Eigenvalues:**

$$\lambda_j = \frac{1}{2}\sum_{k\neq j}^{\mathcal{L}} \frac{z_j^2 + z_k^2}{z_j^2 - z_k^2} - \sum_{i=1}^{N} \frac{z_j^2 + y_i^2}{z_j^2 - y_i^2} - \left(N - \mathcal{L}/2 + \alpha + \beta\right).$$

B. $\xi = \eta^{-1}\gamma^{-1}$, $\zeta = \eta^{-1}\delta^{-1}$: same as from rational *R*-matrix in periodic case!

- **BAE:**

$$-(\gamma+\delta) + \sum_{l=1}^{\mathcal{L}} \frac{1}{y_k^2 - z_l^2} = \sum_{j \neq k}^{N} \frac{2}{y_k^2 - y_j^2}.$$

- **Integrals of motion:**

$$\tau_j = \sum_{k \neq j}^{\mathcal{L}} \frac{2S_k^z S_j^z + S_k^- S_j^+ + S_k^+ S_j^-}{z_j^2 - z_k^2} + (\gamma+\delta) S_j^z.$$

- **Eigenvalues:**

$$\lambda_j = \frac{1}{2} \sum_{k \neq j}^{\mathcal{L}} \frac{1}{z_j^2 - z_k^2} - \sum_{i=1}^{N} \frac{1}{z_j^2 - y_i^2} - \frac{1}{2}(\gamma+\delta).$$

Conclusions and Outlook

Integrals of motion:

$$\text{I:}\quad \tau_j = \sum_{k\neq j}^{\mathcal{L}} \frac{2\left(z_j^2+z_k^2\right)S_j^z S_k^z + 2z_k z_j\left(S_k^- S_j^+ + S_k^+ S_j^-\right)}{z_j^2 - z_k^2} - 2\alpha S_j^z,$$

$$\text{II:}\quad \tau_j = \sum_{k\neq j}^{\mathcal{L}} \frac{2S_k^z S_j^z + S_k^- S_j^+ + S_k^+ S_j^-}{z_j^2 - z_k^2} - 2\alpha S_j^z.$$

	Periodic (QISM) $\rho \to \infty$	**Boundary (BQISM)** $\rho \to 0$
Quasi-classical rat.	II. (s-wave)	I. (case A) or II. (case B)
Quasi-classical trig.	I. (p-wave)	?
Rational	Russian Doll	?
Trigonometric	Anyonic	?

Periodic case: [Dunning, Ibañez, Links, Sierra, Zhao 2010].