# On the quasi-classical limit of the bosonic Lax operator

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# QISM [Faddeev, Kulish, Sklyanin, Takhtajan 1979]

- **Quantum Inverse Scattering Method (QISM)**: a method for construction and solution of **quantum integrable models**.
- **Quantum integrable system** (naive definition): a system that possesses a "complete" set of mutually commuting **conserved operators**.
- **Conserved operator**: an operator that commutes with the Hamiltonian of the system.

**QISM**: ingredients $\Rightarrow$ ... $\Rightarrow$ a set of mutually commuting operators

- Define the Hamiltonian to be a function of these operators $\Rightarrow$ mutually commuting conserved operators $\Rightarrow$ **quantum integrable model**.
- One can construct different models using different types of ingredients.

# Key ingredients

• $R$-**matrix** is an operator $R(u) \in \mathrm{End}(V \otimes V)$ ($V = \mathbb{C}^2$, $u \in \mathbb{C}$) satisfying the **Yang-Baxter Equation** in $\mathrm{End}(V \otimes V \otimes V)$:

$$\boxed{R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v)} \qquad \text{(YBE)}$$

Consider the **trigonometric** $R$-**matrix** ($\eta \in \mathbb{C}$)

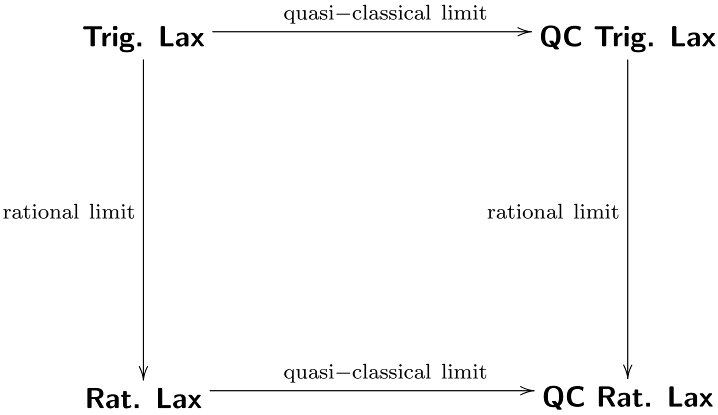
$$R(u) = \frac{1}{\sinh u} \begin{pmatrix} \sinh(u+\eta) & 0 & 0 & 0 \\ 0 & \sinh u & \sinh \eta & 0 \\ 0 & \sinh \eta & \sinh u & 0 \\ 0 & 0 & 0 & \sinh(u+\eta) \end{pmatrix}.$$

• **Lax operator** is an operator $L(u) \in \mathrm{End}(V \otimes W)$ (where $W$ is some vector space) satisfying the **RLL relation** in $\mathrm{End}(V \otimes V \otimes W)$:

$$\boxed{R_{12}(u-v)L_{13}(u)L_{23}(v) = L_{23}(v)L_{13}(u)R_{12}(u-v)} \qquad \text{(RLL)}$$

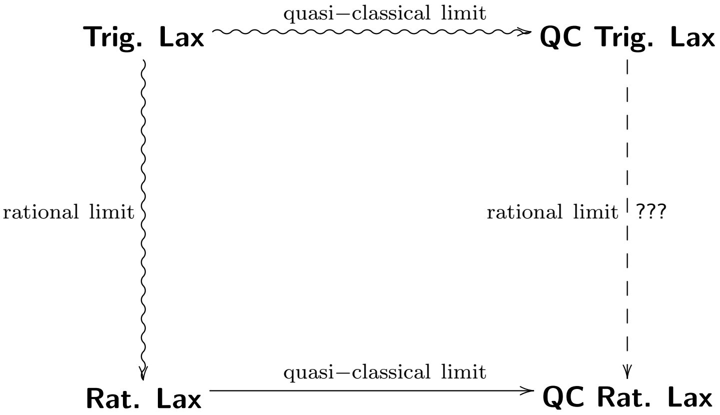
# Outline

For the **spin**-1/2 **Lax operator** ($\dim W = 2$):



# Outline

For the **bosonic Lax operator** ($\dim W = \infty$):



# Spin-1/2

In this case $W = V$ and the **Lax operator** is given by

$$L(u) = \frac{\sinh(u-\eta/2)}{\sinh u} R(u-\eta/2) =$$
$$= \frac{1}{\sinh u}\begin{pmatrix} \sinh u \cosh\frac{\eta}{2} I + 2\cosh u \sinh\frac{\eta}{2} S^z & \sinh\eta S^- \\ \sinh\eta S^+ & \sinh u \cosh\frac{\eta}{2} I - 2\cosh u \sinh\frac{\eta}{2} S^z \end{pmatrix},$$

where $S^+, S^-, S^z$ are the spin-1/2 $\mathfrak{su}(2)$ generators:

$$S^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ S^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ S^z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

They satisfy the $\mathfrak{su}(2)$ commutation relations:

$$[S^z, S^\pm] = \pm S^\pm,\ [S^+, S^-] = 2S^z.$$

- **Rational limit**

$$\frac{\sinh(\nu x)}{\nu} \xrightarrow{\nu\to 0} x, \; \cosh(\nu x) \xrightarrow{\nu\to 0} 1$$

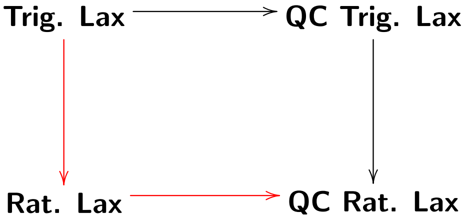
$$L^{rat}(u) = \frac{1}{u}\begin{pmatrix} uI + \eta S^z & \eta S^- \\ \eta S^+ & uI - \eta S^z \end{pmatrix} =$$

$$= I + \frac{\eta}{u}\begin{pmatrix} S^z & S^- \\ S^+ & -S^z \end{pmatrix}$$

- **QC limit of the rational limit**

Coefficient of the $\eta$-term:

$$\ell^{rat}(u) = \frac{1}{u}\begin{pmatrix} S^z & S^- \\ S^+ & -S^z \end{pmatrix}$$



- **Quasi-classical (QC) limit**

Expand as $\eta \to 0$ and take the first non-trivial term in the expansion:

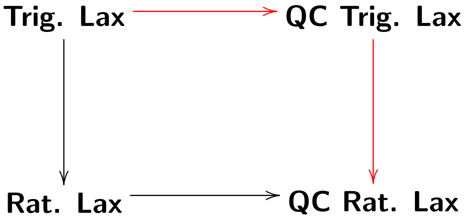
$$L(u) = I + \eta \ell(u) + o(\eta),$$

where

$$\ell(u) = \frac{1}{\sinh u} \begin{pmatrix} \cosh u S^z & S^- \\ S^+ & -\cosh u S^z \end{pmatrix}$$

- **Rational limit of the QC limit**

$$\ell^{rat}(u) = \frac{1}{u} \begin{pmatrix} S^z & S^- \\ S^+ & -S^z \end{pmatrix}$$



# $q$-boson algebra

Make a change of variables $\lambda = e^u$, $q = e^{\eta/2}$. Then the RLL relation

$$\boxed{R_{12}(\lambda/\mu)L_{13}(\lambda)L_{23}(\mu) = L_{23}(\mu)L_{13}(\lambda)R_{12}(\lambda/\mu)} \qquad \text{(RLL)}$$

is satisfied by the following **Lax operator** [Kundu 2007]:

$$L(\lambda) = \begin{pmatrix} \lambda q^{2N+1} - \lambda^{-1} q^{-2N-1} & (q^4 - q^{-4})^{1/2} b_q \\ (q^4 - q^{-4})^{1/2} b_q^\dagger & \lambda q^{-2N-1} + \lambda^{-1} q^{2N+1} \end{pmatrix}, \qquad \text{(L)}$$

with the relations [Macfarlane 1989, Biedenharn 1989]

$$[b_q, b_q^\dagger] = \frac{q^{2(2N+1)} + q^{-2(2N+1)}}{q^2 + q^{-2}}, \; [b_q, N] = b_q, \; [b_q^\dagger, N] = -b_q^\dagger.$$

**Problem:** we can't directly take the rational or the QC limit of (L).

- To be able to take the **rational limit** we need

$$L(\lambda)|_{q=1,\lambda=1} = 0,$$

  but we have $L(\lambda)|_{q=1,\lambda=1} = \text{diag}(0,2)$.
- To be able to take the **QC limit** we need

$$L(\lambda)|_{q=1} \propto I,$$

  but we have $L(\lambda)|_{q=1} = \text{diag}(\lambda - \lambda^{-1}, \lambda + \lambda^{-1})$.

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**Good news:** we can modify (L), so these conditions are met.
This is achieved by rescaling, variable changes and applying a transform

$$L(\lambda) \mapsto AL(\lambda)A, \text{ where } A \text{ is diagonal,}$$

which doesn't violate (RLL).

# Rational limit

Consider the following transform:

$$L(\lambda) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & (q^4 - q^{-4})^{1/2} \end{pmatrix} L(\lambda) \begin{pmatrix} 1 & 0 \\ 0 & (q^4 - q^{-4})^{1/2} \end{pmatrix}.$$

The **modified Lax operator**, which satisfies (RLL), is of the form

$$L(\lambda)' = \begin{pmatrix} \lambda q^{2N+1} - \lambda^{-1} q^{-2N-1} & (q^4 - q^{-4}) b_q \\ (q^4 - q^{-4}) b_q^\dagger & (q^4 - q^{-4})(\lambda q^{-2N-1} + \lambda^{-1} q^{2N+1}) \end{pmatrix}. \quad \text{(L}'\text{)}$$

The **rational limit** of (L′) gives

$$L^{rat}(u)' = \begin{pmatrix} 2u + \eta(2N+1) & 4\eta b \\ 4\eta b^\dagger & 8\eta \end{pmatrix}. \quad \text{(L-rat}'\text{)}$$

**Now**: to take the QC limit we want $L^{rat}(u)'|_{\eta=0} \propto I$.

A transform

$$L^{rat}(u)' \mapsto \frac{1}{4}\begin{pmatrix}(2\eta)^{1/2} & 0 \\ 0 & (2\eta)^{-1/2}\end{pmatrix} L^{rat}(u)' \begin{pmatrix}(2\eta)^{1/2} & 0 \\ 0 & (2\eta)^{-1/2}\end{pmatrix}$$

together with a variable change $u \mapsto u - \eta/2 + \eta^{-1}$ gives

$$L^{rat}(u) = \begin{pmatrix}(1+\eta u)I + \eta^2 N & \eta b \\ \eta b^\dagger & I\end{pmatrix}. \qquad \text{(L-rat)}$$

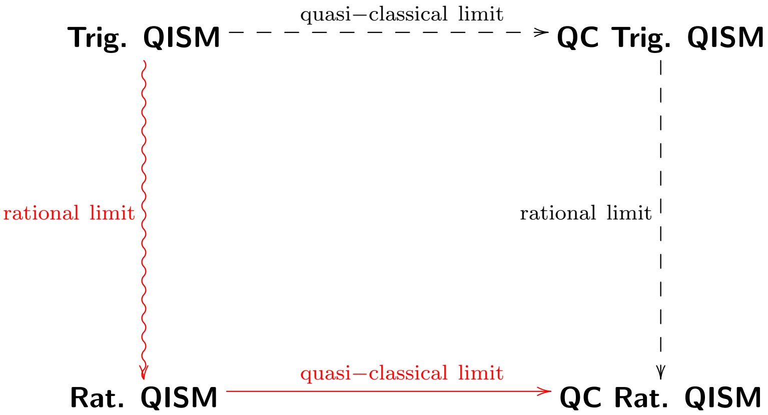
The **QC limit of the rational limit**:

$$L^{rat}(u) = I + \eta \ell^{rat}(u) + o(\eta^2),$$

where

$$\ell^{rat}(u) = \begin{pmatrix} u & b \\ b^\dagger & 0\end{pmatrix}. \qquad \text{(L-rat-qc)}$$

# Starting with the rational limit



## Quasi-classical limit

Consider the mapping $L(\lambda) \mapsto (q^4 - q^{-4})^{1/2} L(\lambda)$ together with the change of variable $(q^4 - q^{-4})^{1/2}\lambda \mapsto \lambda$.

The **modified Lax operator**:

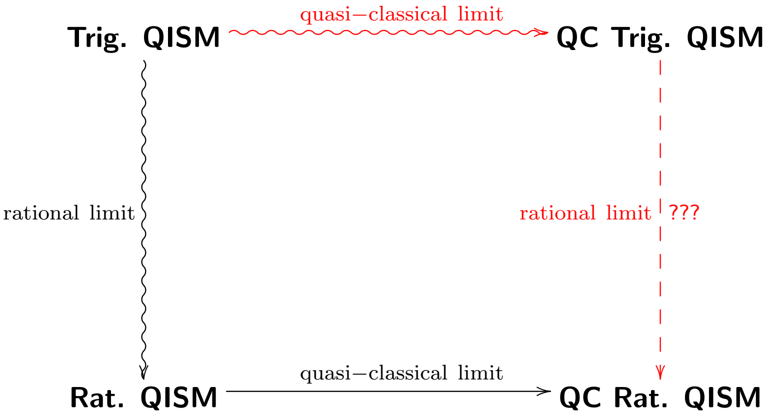
$$L(\lambda)'' = \begin{pmatrix} \lambda q^{2N+1} - (q^4 - q^{-4})\lambda^{-1} q^{-2N-1} & (q^4 - q^{-4}) b_q \\ (q^4 - q^{-4}) b_q^\dagger & \lambda q^{-2N-1} + (q^4 - q^{-4})\lambda^{-1} q^{2N+1} \end{pmatrix}. \tag{L''}$$

In the **QC limit** (with a variable change $\lambda \mapsto 2\sqrt{2}\lambda$) we obtain

$$\ell(\lambda) = \frac{1}{\lambda} \begin{pmatrix} \lambda N + 1/2(\lambda - \lambda^{-1}) & \sqrt{2} b \\ \sqrt{2} b^\dagger & -(\lambda N + 1/2(\lambda - \lambda^{-1})) \end{pmatrix}. \tag{L-qc}$$

**Note:** $\ell(1) = \begin{pmatrix} N & \sqrt{2}b \\ \sqrt{2}b^\dagger & -N \end{pmatrix} \neq 0$, so this operator doesn't have an obvious rational limit.

# Starting with the QC limit



# An observation

Consider the following operator in $\mathrm{End}(V \otimes W \otimes W)$:

$$T(\lambda) = L_1(\lambda) L_2(i\lambda).$$

It satisfies the RLL type relation in $\mathrm{End}(V \otimes V \otimes (W \otimes W))$:

$$\boxed{R_{12}(\lambda/\mu) T_{13}(\lambda) T_{23}(\mu) = T_{23}(\mu) T_{13}(\lambda) R_{12}(\lambda/\mu)}$$

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \text{ with } A(\lambda), B(\lambda), C(\lambda), D(\lambda) \in \mathrm{End}(W \otimes W).$$

**Observation:** $T(\lambda)$ satisfies both conditions needed for taking the rational and the QC limits, i.e.

$$T(\lambda)|_{q=1} \propto I \text{ and } T(\lambda)|_{q=1,\lambda=1} = 0.$$

In the **rational limit** we obtain

$$
\begin{aligned}
A^{rat}(u) =& 4iu + 4\eta\left[i(N_1+1/2)+b_1 b_2^\dagger\right],\\
B^{rat}(u) =& 4\eta^{1/2}u\left[ib_1+b_2\right]+4\eta^{3/2}\left[(N_1+1/2)b_2 - ib_1(N_2+1/2)\right],\\
C^{rat}(u) =& 4\eta^{1/2}\left[ib_1^\dagger + b_2^\dagger\right],\\
D^{rat}(u) =& 4iu - 4\eta\left[i(N_2+1/2)-b_1^\dagger b_2\right].
\end{aligned}
$$

**Note:** we have $T^{rat}(u)|_{\eta=0} = 4iuI$, but what do we do with $\eta^{1/2}$?
In the QC expansion of the **rational** $R$**-matrix** $\eta$ is the leading power!

$$R^{rat}(u) = I + \frac{\eta}{u}P,$$

where $P$ is the permutation operator: $P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$

# Conclusions

- The bosonic case is not as straightforward as the spin-1/2 case.
- The fact that the rational and the QC limits in general **don't commute** opens possibilities to investigate wider range of applications, in particular, **bosonic tunneling models**:
  - [Enol'skii, Kuznetsov, Salerno 1993],
  - [Links, Hibberd 2006],
  - [Links, Foerster, Tonel, Santos 2006],
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Thank you for your attention!