Internal Time and Quantum Action Principle in Relativistic Quantum Mechanics

Inna Lukyanenko, UQ
Alexander Lukyanenko,
St. Petersburg State Polytechnical University

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THE UNIVERSITY OF QUEENSLAND
AUSTRALIA

Outline

- canonical **quantization procedure**
- new approach based on an **internal time** parameter and **quantum action** principle
- allows a **probabilistic interpretation**
- **non-relativistic limit** of the theory

Non-relativistic particle in $\mathbb{R}^3$

Coordinate representation in the state space $L^2(\mathbb{R}^3)$:

$$\begin{cases} \hat{x}_j \psi(x) = x_j \psi(x), \\ \hat{p}_j \psi(x) = \frac{\hbar}{i} \frac{\partial}{\partial x_j} \psi(x). \end{cases}$$

The dynamics is governed by the **Schrödinger equation**

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi(x,t) + V(x)\psi(x,t).$$

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$$\frac{\partial |\psi|^2}{\partial t} + \mathrm{div}\mathbf{j} = 0, \text{ where } \mathbf{j} = \frac{i\hbar}{2m}(\psi \nabla \bar{\psi} - \bar{\psi} \nabla \psi).$$

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But: this equation is not symmetric in t and x!

Relativistic particle

Coordinates $x^\mu \equiv (ct, x^1, x^2, x^3)$ in the Minkowsky space $\mathbb{R}^{1,3}$.
Minkowsky metric: $\eta_{\mu\nu} = diag(1, -1, -1, -1)$; $\mu, \nu = 0, 1, 2, 3$.
Action Integral on a world line $x^\mu(\tau),\ \tau \in [0, 1]$:

$$\mathcal{I}[x(\tau)] = \int_0^1 L(x, \dot{x}) d\tau = -mc \int_0^1 \sqrt{\dot{x}^2} d\tau, \quad \dot{x}^2 \equiv \eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$$

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Lagrangian: $L(x, \dot{x}) = -mc\sqrt{\dot{x}^2} \;\Rightarrow\; p_\mu \equiv \dfrac{\partial L(x, \dot{x})}{\partial \dot{x}^\mu} = -mc \dfrac{\dot{x}_\mu}{\sqrt{\dot{x}^2}}$.

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Instead we get a **constraint** for the canonical variables:

$$\boxed{H \equiv p_\mu p^\mu - m^2 c^2 = 0}$$

Canonical form of the action (method of Lagrange multipliers):

$$\mathcal{I}[x, p] = \int_0^1 (p_\mu \dot{x}^\mu - NH) d\tau$$

Canonical Quantization (Covariant Method)

Constraint operator: $\widehat{H} \equiv H(\hat{x}, \hat{p}) = -\hbar^2 \nabla_\mu \nabla^\mu - m^2 c^2$, where $\nabla_\mu \nabla^\mu \psi = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} - \Delta \psi$ is the **D'Alembert operator**.

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It is relativistic, but there is **no probabilistic interpretation**:

- $|\psi(ct, x)|^2$ is not conserved: $\frac{d}{dt} \int |\psi(ct, x)|^2 d^3 x \neq 0$.
- $J^0 \equiv \frac{i\hbar}{c} \left(\bar{\psi} \frac{\partial \psi}{\partial t} - \psi \frac{\partial \bar{\psi}}{\partial t} \right)$ obeys $\frac{\partial J^0}{\partial t} = \mathrm{div}\mathbf{j}$, where $\mathbf{j} = i\hbar c \left(\bar{\psi} \nabla^k \psi - \psi \nabla^k \bar{\psi} \right)$. **But:** it is not positive definite!

Relativistic Schrödinger Equation

V. Fock: introduce an **internal time** parameter $s \in [0, C]$, where $C = \int_0^1 N(\tau) d\tau$, and write a **relativistic Schrödinger eq. (RSE)**:

$$\boxed{i\hbar \frac{\partial \psi(s, x^\mu)}{\partial s} = \widehat{H} \psi(s, x^\mu)}$$

Canonical action: $\mathcal{I}[x, p] = \int_0^C (p_\mu \dot{x}^\mu - H) ds, \quad ds = N(\tau) d\tau$

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Our proposal: to connect $s \in [0, C]$ with a certain experiment ($s = 0$ is the beginning, $s = C$ the end of experiment).

Experiment

A particle is emitted somewhere in a space-time domain $\Omega_0 \subset \mathbb{R}^{1,3}$.
Let $\psi_0(x^\mu) \equiv \psi(0, x^\mu)$ be the **initial state** of the particle, s.t.

$$\int_{\Omega_0} |\psi_0(x^\mu)|^2 d^4 x^\mu < \infty.$$

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Let the state develop according to **RSE** up to the moment $s = C$, when the particle is detected: $\psi_0(x^\mu) \rightarrow \psi(C, x^\mu)$.

Then $|\psi(C, x_1^\mu)|^2$ can be interpreted as the **probability density** to detect the particle near the space-time point x_1^μ.

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Problem: how to fix the internal time parameter C?

Quantum Action Principle

We propose a **Quantum Action Principle (QAP)**. Let

$$\psi(C, x_1^\mu) = R(C, x_1^\mu) \exp\left[\frac{i}{\hbar} S(C, x_1^\mu)\right].$$

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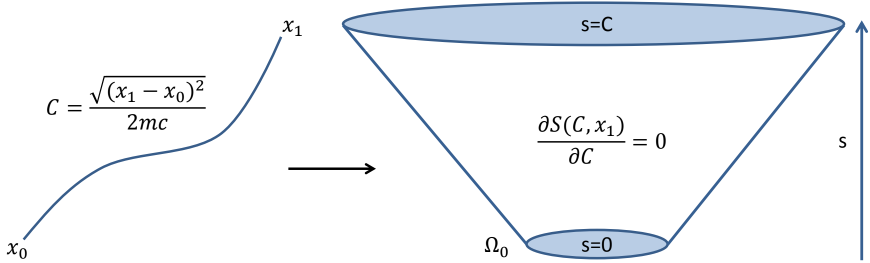
We will take it as a **quantum action**. The **stationarity condition** of the quantum action:

$$\boxed{\frac{\partial S(C, x_1)}{\partial C} = 0}$$

A stationary solution C_{ext} of QAP will be a function of the end point x_1^μ and of the initial state $\psi_0(x^\mu)$ of the particle.

$$C_{ext} = C_{ext}(x_1^\mu)$$

Picture


x_1
$C = \frac{\sqrt{(x_1 - x_0)^2}}{2mc}$
x_0
s=C
$\frac{\partial S(C, x_1)}{\partial C} = 0$
s
Ω_0
s=0

Probabilistic Interpretation

Substituting $C_{ext}(x_1^\mu)$ in the solution, we obtain the **probability density** to detect the particle near the point x_1^μ of the Minkowsky space (time t_1 is also a stochastic parameter):

$$\rho_{ext}(x_1^\mu) \equiv |\psi(C_{ext}(x_1^\mu), x_1^\mu)|^2.$$

Taking into account all possible outcomes of the experiment we get a function $\rho_{ext}(x^\mu)$ on the Minkowsky space.

Normalization: doesn't follow directly from RSE, must accord with the experiment. We impose a **normalization condition:**

$$\int_0^\infty \int_\Sigma \rho_{ext}(x^\mu) dx^0 d^2\sigma = 1,$$

i.e. a particle will be detected with the probability 1.

Non-relativistic Limit

Take an initial state, where t is definite: $\psi_0(x^\mu) = \delta(t)\psi_0'(x^k)$.
Proposition. *In the non-relativistic limit, when the stationary value of the internal time is* $C_{ext} = \frac{t}{2m}$, *the solution of the RSE is*

$$\psi(x^\mu) = \exp\left(-\frac{i}{\hbar}mc^2 t\right)\psi'(t, x^k), \text{ where}$$

$\psi'(t, x^k)$ *satisfies the Schrödinger equation:* $i\hbar\frac{\partial\psi'}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi'$.

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$\psi'(t, x^k)$ *satisfies the Schrödinger equation:* $i\hbar\frac{\partial\psi'}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi'$.

Time t becomes a classical parameter, and the probability density

$$\rho(t, x^k) = |\psi'(t, x^k)|^2$$

is the probability density of a particle to be detected in the point x^k at the moment of time t.

Conclusions and Outlook

- use of the **internal time** and **quantum action principle** for quantization of 1-particle relativistic mechanics
- allows a **probabilistic interpretation** and gives the proper **non-relativistic limit**
- **Next step:** application to more complicated systems (QFT, General Relativity)

Thank you for your attention!

N. N. Gorobey, A. S. Lukyanenko, I. A. Lukyanenko, *Quantum Action Principle in Relativistic Mechanics (II)*, arXiv:1010.3824vl [quant-ph] 19 Oct 2010.

N. N. Gorobey, A. S. Lukyanenko, I. A. Lukyanenko, *On a Probabilistic Interpretation of Relativistic Quantum Mechanics*, arXiv:1012.1719vl [quant-ph] 8 Dec 2010.

e-mail: alex.lukyan@rambler.ru