Integrable models from the Boundary Quantum Inverse Scattering Method

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History

- Quantum Inverse Scattering Method (QISM) for periodic boundary conditions [Faddeev 1979]
- QISM for open-boundary conditions [Sklyanin 1987]

R-matrix

R-**matrix** is an operator $R(u) \in \mathrm{End}(V \otimes V)$ ($V \cong \mathbb{C}^2$, $u \in \mathbb{C}$) satisfying the **Yang-Baxter equation** in $\mathrm{End}(V \otimes V \otimes V)$:

$$\boxed{R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v).}$$

I. Trigonometric solution

$$R^{trig}(u) = \frac{1}{\sinh(u+\eta)} \begin{pmatrix} \sinh(u+\eta) & 0 & 0 & 0 \\ 0 & \sinh u & \sinh\eta & 0 \\ 0 & \sinh\eta & \sinh u & 0 \\ 0 & 0 & 0 & \sinh(u+\eta) \end{pmatrix}.$$

II. Rational solution

$$R^{rat}(u) = \frac{1}{u+\eta}(uI \otimes I + \eta P) = \frac{1}{u+\eta} \begin{pmatrix} u+\eta & 0 & 0 & 0 \\ 0 & u & \eta & 0 \\ 0 & \eta & u & 0 \\ 0 & 0 & 0 & u+\eta \end{pmatrix},$$

QISM

Monodromy matrix (acts in $\mathrm{End}(V_a \otimes V^{\otimes \mathcal{L}})$, $V_a = \mathbb{C}^2$ auxiliary space)

$$T_a(u) \equiv \begin{pmatrix} e^{-\eta\alpha} & 0 \\ 0 & e^{\eta\alpha} \end{pmatrix} R_{a\mathcal{L}}(u-\varepsilon_{\mathcal{L}})\ldots R_{a2}(u-\varepsilon_2)R_{a1}(u-\varepsilon_1) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

satisfies the Yang-Baxter equation in $\mathrm{End}(V_a \otimes V_b \otimes V^{\otimes \mathcal{L}})$:

$$R_{ab}(u-v)T_a(u)T_b(v) = T_b(v)T_a(u)R_{ab}(u-v).$$

Transfer matrix $t(u) \equiv tr_a(T_a(u)) = A(u) + D(u) \in \mathrm{End}(V^{\otimes \mathcal{L}})$,

$$\boxed{[t(u), t(v)] = 0 \quad \forall u, v \in \mathbb{C}.}$$

Consider the expansion of $t(u)$ in powers of u:

$$t(u) = \sum_{j=-\infty}^{\infty} C_j u^j.$$

Then C_j form a set of mutually commuting **integrals of motion**.

Bethe Ansatz

Start with a **reference state**, $\Omega \in V$, such that

$$B(u)\Omega = 0, \; A(u)\Omega = a(u)\Omega, \; D(u)\Omega = d(u)\Omega, \; C(u)\Omega \neq 0,$$

Then (for the **trigonometric** R-matrix)

$$\Phi(v_1, ..., v_N) = C(v_1)...C(v_N)\Omega$$

is an **eigenstate** of $t(u)$ with the **eigenvalue**

$$\Lambda(u, v_1, ..., v_N) = a(u) \prod_{k=1}^{N} \frac{\sinh(u - v_k + \eta)}{\sinh(u - v_k)} + d(u) \prod_{k=1}^{N} \frac{\sinh(u - v_k - \eta)}{\sinh(u - v_k)},$$

if $\Phi \neq 0$ and v's satisfy the **Bethe Ansatz equations**

$$\frac{a(v_k)}{d(v_k)} = \prod_{j \neq k}^{N} \frac{\sinh(v_k - v_j - \eta)}{\sinh(v_k - v_j + \eta)}, \quad k = 1, ..., N.$$

For the **rational** R-matrix: $\sinh(x) \to x$.

Quasi-classical limit

Now we take the quiasi-classical limit: $\eta \to 0$.
Spin operators

$$S^+ = S^x + iS^y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ S^- = S^x - iS^y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$S^x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ S^y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ S^z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

I. Trigonometric case

- **BAE:**

$$(\alpha + N - 1 - \mathcal{L}/2) + \sum_{l=1}^{\mathcal{L}} \frac{y_k^2}{y_k^2 - y_j^2} = 2 \sum_{j \neq k}^{N} \frac{y_k^2}{y_k^2 - y_j^2}.$$

- **Integrals of motion:**

$$T_j = 2 \sum_{k \neq j}^{\mathcal{L}} \frac{z_j^2 + z_k^2}{z_j^2 - z_k^2} S_j^z S_k^z + \sum_{k \neq j}^{\mathcal{L}} \frac{2 z_k z_j}{z_j^2 - z_k^2} \left(S_k^- S_j^+ + S_k^+ S_j^- \right) - 2\alpha S_j^z.$$

- **Eigenvalues:**

$$\Lambda_j = \alpha + \frac{1}{2} \sum_{k \neq j}^{\mathcal{L}} \frac{z_j^2 + z_k^2}{z_j^2 - z_k^2} - \sum_{i=1}^{N} \frac{z_j^2 + y_i^2}{z_j^2 - y_i^2}.$$

II. Rational case

- **BAE:**

$$\alpha + \frac{1}{2}\sum_{l=1}^{\mathcal{L}} \frac{1}{v_k - \varepsilon_l} = \sum_{j \neq k}^{N} \frac{1}{v_k - v_j},$$

- **Integrals of motion:**

$$T_j = \sum_{k \neq j}^{\mathcal{L}} \frac{2S_k^z S_j^z + S_k^- S_j^+ + S_k^+ S_j^-}{\varepsilon_j - \varepsilon_k} - 2\alpha S_j^z,$$

- **Eigenvalues:**

$$\Lambda_j = \alpha + \frac{1}{2}\sum_{k \neq j}^{\mathcal{L}} \frac{1}{\varepsilon_j - \varepsilon_k} - \sum_{i=1}^{N} \frac{1}{\varepsilon_j - v_i}.$$

Reflection equations

Start with a **rational** R-matrix $R(u) = \frac{1}{u+\eta}(uI \otimes I + \eta P)$.

In addition to YBE we want it to satisfy **reflection equations**:

$$\begin{cases} R_{12}(u-v)K_1^-(u)R_{21}(u+v)K_2^-(v) = K_2^-(v)R_{12}(u+v)K_1^-(u)R_{21}(u-v), \\ R_{12}(v-u)K_1^+(u)\mathcal{R}_{21}(u+v)K_2^+(v) = K_2^+(v)\mathcal{R}_{12}(u+v)K_1^+(u)R_{21}(v-u), \end{cases}$$

where $\mathcal{R}(u) \equiv R(-u-2\eta)$. Easy to check that

$$K^-(u) = \zeta I + u\sigma^z = \begin{pmatrix} \zeta + u & 0 \\ 0 & \zeta - u \end{pmatrix},$$

$$K^+(u) = \xi I + (u+\eta)\sigma^z = \begin{pmatrix} \xi + u + \eta & 0 \\ 0 & \xi - u - \eta \end{pmatrix}.$$

satisfy these equations. They are called the **reflection matrices**.

BQISM

Define the **double monodromy matrix** (acting in $V_a \otimes V^{\otimes \mathcal{L}}$)

$$T_a(u) \equiv R_{a\mathcal{L}}(u-\varepsilon_{\mathcal{L}})\dots R_{a1}(u-\varepsilon_1)K_a^-(u+\rho/2)\times$$
$$\times R_{a1}^{-1}(-u-\varepsilon_1-\rho)\dots R_{a\mathcal{L}}^{-1}(-u-\varepsilon_{\mathcal{L}}-\rho) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}.$$

It satisfies the Yang-Baxter equation in $V_a \otimes V_b \otimes V^{\otimes \mathcal{L}}$:

$$R_{ab}(u-v)T_a(u)R_{ba}(u+v+\rho)T_b(v) = T_b(v)R_{ab}(u+v+\rho)T_a(u)R_{ba}(u-v).$$

Define the **transfer matrix**

$$t(u) \equiv tr_a\left(K_a^+(u+\rho/2)T_a(u)\right) = (\xi+u+\rho/2+\eta)A(u)+(\xi-u-\rho/2-\eta)D(u).$$

In this case one can also show that

$$[t(u), t(v)] = 0 \ \ \forall u, v \in \mathbb{C}.$$

Bethe Ansatz equations

Shift $u \to u - \frac{\eta}{2}$ and introduce $\tilde{A}(u) = (2u+\rho)A(u) - \eta D(u)$.

$$\Phi(v_1, \ldots, v_N) = C(v_1)\ldots C(v_N)\Omega$$

is an **eigenstate** of $t(u)$ with the **eigenvalue**

$$\Lambda(u, v_1, \ldots, v_N) = \tilde{a}(u)\frac{\xi + u + \rho/2 + \eta/2}{2u+\rho}\prod_{k=1}^{N}\frac{(u - v_k + \eta)(u + v_k + \rho + \eta)}{(u - v_k)(u + v_k + \rho)} +$$

$$+ d(u)\frac{(2u+\rho+\eta)(\xi - u - \rho/2 + \eta/2)}{2u+\rho}\prod_{k=1}^{N}\frac{(u - v_k - \eta)(u + v_k + \rho - \eta)}{(u - v_k)(u + v_k + \rho)},$$

if $\Phi \neq 0$ and v's satisfy the **Bethe Ansatz equations (BAE)**

$$\frac{\tilde{a}(v_k)}{d(v_k)(2v_k + \rho - \eta)}\frac{\xi + v_k + \rho/2 + \eta/2}{\xi - v_k - \rho/2 + \eta/2} = \prod_{j \neq k}^{N}\frac{(v_k - v_j - \eta)(v_k + v_j + \rho - \eta)}{(v_k - v_j + \eta)(v_k + v_j + \rho + \eta)}.$$

Can **renormalize**, so that it goes to non-boundary case as $\rho \to \infty$.

Quasi-classical limit

Consider **two limiting cases:**

1. $\xi = \eta\alpha, \quad \zeta = \eta\beta.$
2. $\xi = \eta^{-1}\gamma^{-1}, \quad \zeta = \eta^{-1}\delta^{-1}.$

Notation: $y_k = v_k + \dfrac{\rho}{2}, \; z_l = \varepsilon_l + \dfrac{\rho}{2}.$

Case 1: $\xi = \eta\alpha, \; \zeta = \eta\beta$

- **BAE:**

$$-(\alpha+\beta+1) + \sum_{l=1}^{\mathcal{L}} \frac{y_k^2}{y_k^2 - z_l^2} = \sum_{j\neq k}^{N} \frac{2y_k^2}{y_k^2 - y_j^2}.$$

- **Integrals of motion:**

$$T_j = \sum_{k\neq j}^{\mathcal{L}} \frac{z_j^2 + z_k^2}{z_j^2 - z_k^2} 2S_j^z S_k^z + \sum_{k\neq j}^{\mathcal{L}} \frac{2z_j z_k}{z_j^2 - z_k^2}\left(S_k^- S_j^+ + S_k^+ S_j^-\right) + \\ + \left(2N - \mathcal{L} + 2(\alpha+\beta)\right) N_j - N.$$

- **Eigenvalues:**

$$\Lambda_j = -\sum_{i=1}^{N} \frac{2z_j^2}{z_j^2 - y_i^2}.$$

Case 2: $\xi = \eta^{-1}\gamma^{-1}$, $\zeta = \eta^{-1}\delta^{-1}$

- **BAE:**

$$-(\delta+\gamma) + \sum_{l=1}^{\mathcal{L}} \frac{1}{y_k^2 - z_l^2} = \sum_{j\neq k}^{N} \frac{2}{y_k^2 - y_j^2}.$$

- **Integrals of motion:**

$$T_j = \sum_{k\neq j}^{\mathcal{L}} \frac{2P_{kj}}{z_j^2 - z_k^2} + (\gamma+\delta)\sigma_j^z.$$

- **Eigenvalues:**

$$\Lambda_j = -\sum_{i=1}^{N} \frac{1}{z_j^2 - y_i^2} - (\gamma+\delta).$$

Conclusions and Outlook

	Periodic	Boundary (1)	Boundary (2)	Boundary (general)
Rat.	s-wave	s-wave	p+ip-wave	?
Trig.	p+pi-wave	p+ip-wave(?)	?	??

Thank you for your attention!

[1] E.K. Sklyanin, J. Phys. A: Math. Gen. **21**, 2375 (1988).
[2] T. Skrypnyk, J. Stat. Mech.: Theor. Exp. P06028 (2010).