Properties of the Bethe Ansatz equations for Richardson-Gaudin models

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Some History

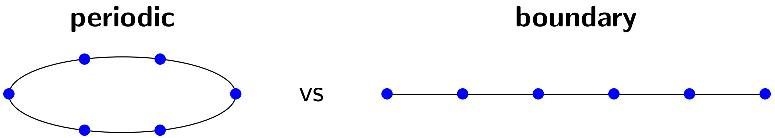
- **Quantum Inverse Scattering Method (QISM)** for twisted periodic boundary conditions [Faddeev, Kulish, Sklyanin, Takhtajan 1979].

- **BCS model** of superconductivity [Bardeen, Cooper and Schrieffer 1957].
  - Solved for the **rational case** [Richardson 1963].
  - **Integrals of motion** for the Richardson model [Cambiaggio et al. 1997].
  - **Eigenvalues** for the Richardson model [Sierra 2000].
  - Generalized to the **trigonometric case** using Gaudin's method [Amico et al. 2001], [Dukelsky et al. 2001].
  - Reformulated through the **quasi-classical limit** of QISM [Zhou et al. 2002], [von Delft and Poghossian 2002].

- **Some extensions:** [Ovchinnikov 2003], [Dunning and Links 2004], [Ibañez et al. 2009], [Skrypnyk 2009], [Dukelsky et al. 2010, 2011], [Links and Marquette 2013].

Some History

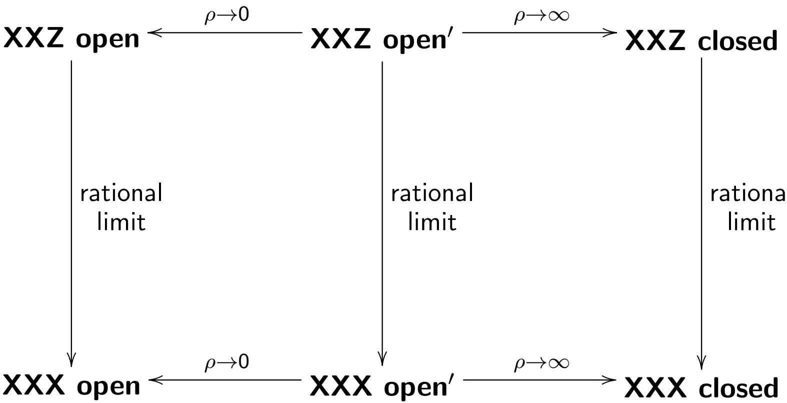
- **Boundary QISM (BQISM)** for open-boundary conditions, for the case of XXZ spin chain [Sklyanin 1988].



- What is the effect of the "boundary" for **Richardson-Gaudin models**?
  - **Gaudin magnet** with boundary [Hikami 1995].
  - **Quasi-classical limit** of the BQISM [Di Lorenzo et al. 2002].
  - **Generalized** Gaudin systems [Skrypnyk 2006, 2007, 2010].
  - **Trigonometric** Gaudin model [Cirilio António et al. 2013].

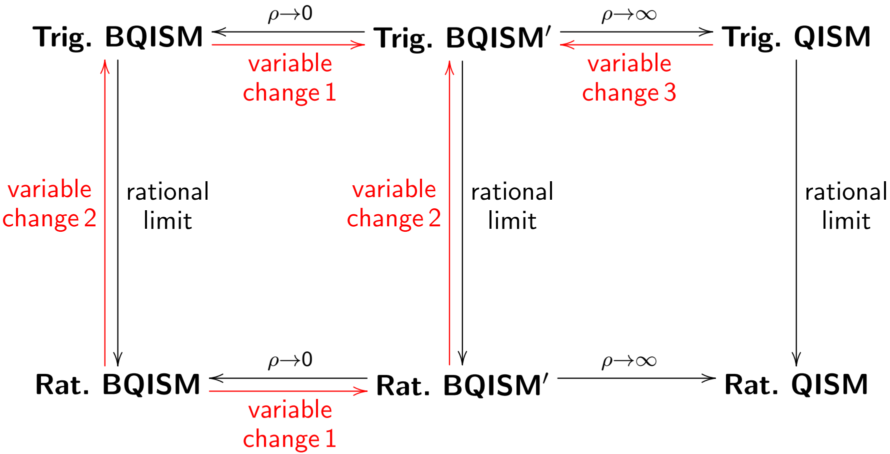
Outline

For the **Heisenberg model**



Outline

For the **Richardson-Gaudin models**



R-matrix

R-**matrix** is an operator $R(u) \in \mathrm{End}(V \otimes V)$ $(V = \mathbb{C}^2,\ u \in \mathbb{C})$ satisfying the **Yang-Baxter equation (YBE)** in $\mathrm{End}(V \otimes V \otimes V)$:

$$\boxed{R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v)}$$

- **Rational solution** $(\eta \in \mathbb{C},\quad P(u \otimes v) = v \otimes u,\ \forall u, v \in V)$

$$R^{rat}(u) = \frac{1}{u+\eta}(uI \otimes I + \eta P) = \frac{1}{u+\eta}\begin{pmatrix} u+\eta & 0 & 0 & 0 \\ 0 & u & \eta & 0 \\ 0 & \eta & u & 0 \\ 0 & 0 & 0 & u+\eta \end{pmatrix}.$$

- **Trigonometric solution**

$$R^{trig}(u) = \frac{1}{\sinh(u+\eta)}\begin{pmatrix} \sinh(u+\eta) & 0 & 0 & 0 \\ 0 & \sinh u & \sinh\eta & 0 \\ 0 & \sinh\eta & \sinh u & 0 \\ 0 & 0 & 0 & \sinh(u+\eta) \end{pmatrix}.$$

Rational limit $\lim_{\nu \to 0} \frac{\sinh(\nu x)}{\nu} = x$: $\boxed{\textbf{Trigonometric} \longrightarrow \textbf{Rational}}$

QISM [Faddeev et al. 1979]

Monodromy matrix $\in \text{End}\left(V_a \otimes V^{\otimes \mathcal{L}}\right)$ (where $V_a = \mathbb{C}^2$ is the auxiliary space, $V^{\otimes \mathcal{L}} = \underbrace{V \otimes V \otimes \ldots \otimes V}_{\mathcal{L} \text{ times}}$ is the quantum space, $\mathcal{L} \in \mathbb{N}$)

$$T_a(u) = \begin{pmatrix} e^{-\eta\gamma} & 0 \\ 0 & e^{\eta\gamma} \end{pmatrix} R_{a\mathcal{L}}(u - \varepsilon_{\mathcal{L}}) \ldots R_{a2}(u - \varepsilon_2) R_{a1}(u - \varepsilon_1) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}.$$

Transfer matrix $t(u) = tr_a(T_a(u)) = A(u) + D(u) \in \text{End}(V^{\otimes \mathcal{L}})$:

$$\boxed{[t(u), t(v)] = 0 \quad \forall u, v \in \mathbb{C}}$$

Then $t(u)$ generates a set of **mutually commuting** operators $\{C_j\}$:

$$t(u) = \sum_{j=-\infty}^{\infty} C_j u^j.$$

Take any function of $\{C_j\}$ as the Hamiltonian. Then $\{C_j\}$ are mutually commuting **integrals of motion**.

Algebraic Bethe Ansatz

Start with a **reference state** $\Omega \in V^{\otimes \mathcal{L}}$:

$$B(u)\Omega = 0, \; A(u)\Omega = a(u)\Omega, \; D(u)\Omega = d(u)\Omega, \; C(u)\Omega \neq 0.$$

Then (for the **trigonometric** R-matrix)

$$\Phi(v_1, ..., v_N) = C(v_1)...C(v_N)\Omega$$

is an **eigenstate** of $t(u)$ with the **eigenvalue**

$$\Lambda(u, v_1, ..., v_N) = a(u) \prod_{k=1}^{N} \frac{\sinh(u - v_k + \eta)}{\sinh(u - v_k)} + d(u) \prod_{k=1}^{N} \frac{\sinh(u - v_k - \eta)}{\sinh(u - v_k)},$$

if $\Phi \neq 0$ and v's satisfy the **Bethe Ansatz equations (BAE)**

$$\boxed{\frac{a(v_k)}{d(v_k)} = \prod_{i \neq k}^{N} \frac{\sinh(v_k - v_i - \eta)}{\sinh(v_k - v_i + \eta)}, \quad k = 1, ..., N}$$

For the **rational** R-matrix: $\sinh(x) \to x$.

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For the **rational** R-matrix: $\sinh(x) \to x$.

Quasi-classical limit

Derive the expressions for $a(u)$ and $d(u)$ for $\Omega = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{\otimes \mathcal{L}}$:

$$a(u) = e^{-\eta\gamma} \prod_{l=1}^{\mathcal{L}} \frac{\sinh(u - \varepsilon_l - \eta/2)}{\sinh(u - \varepsilon_l)}, \quad d(u) = e^{\eta\gamma} \prod_{l=1}^{\mathcal{L}} \frac{\sinh(u - \varepsilon_l + \eta/2)}{\sinh(u - \varepsilon_l)}.$$

Then the **BAE**:

$$e^{-2\eta\gamma} \prod_{l=1}^{\mathcal{L}} \frac{\sinh(v_k - \varepsilon_l - \eta/2)}{\sinh(v_k - \varepsilon_l + \eta/2)} = \prod_{i \neq k}^{N} \frac{\sinh(v_k - v_i - \eta)}{\sinh(v_k - v_i + \eta)}.$$

Take first non-zero term as $\eta \to 0$ to obtain the **quasi-classical limit** of the QISM.

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Take first non-zero term as $\eta \to 0$ to obtain the **quasi-classical limit** of the QISM.

- **Trig. QISM [Amico et al. 2001], [Dukelsky et al. 2001]**

$$\boxed{2\gamma + \sum_{l=1}^{\mathcal{L}} \coth(v_k - \varepsilon_l) = 2\sum_{i\neq k}^{N} \coth(v_k - v_i)} \qquad (\diamondsuit)$$

- **Rat. QISM [Richardson 1963]**

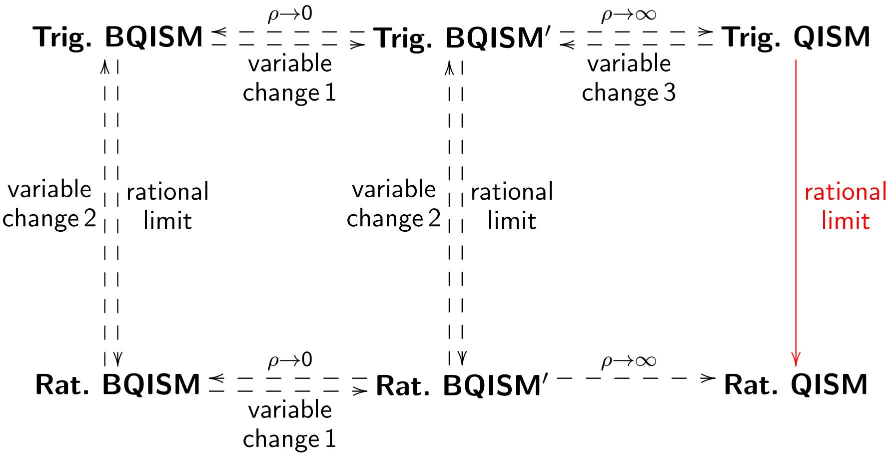
$$\boxed{2\gamma + \sum_{l=1}^{\mathcal{L}} \frac{1}{v_k - \varepsilon_l} = \sum_{i\neq k}^{N} \frac{2}{v_k - v_i}} \qquad (\heartsuit)$$

$$\boxed{\textbf{Trig. QISM } (\diamondsuit) \xrightarrow{\text{rational limit}} \textbf{Rat. QISM } (\heartsuit)}$$

- Change of variables $v_k \mapsto \ln y_k,\ \varepsilon_l \mapsto \ln z_l$ in $(\diamondsuit)$ gives

$$\boxed{(N - \mathcal{L}/2 + \gamma - 1) + \sum_{l=1}^{\mathcal{L}} \frac{y_k^2}{y_k^2 - z_l^2} = \sum_{i\neq k}^{N} \frac{2y_k^2}{y_k^2 - y_i^2}}$$

Diagram


Trig. BQISM
ρ→0
variable change 1
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ρ→∞
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ρ→∞
Rat. QISM

Reflection equations [Cherednik 1984]

Start with the **trigonometric** R-matrix. In addition to the YBE we require that it satisfies the **reflection equations** for some $K^{\pm} \in \mathrm{End}(V)$, referred to as the **reflection matrices**:

$$\begin{cases} R_{12}(u-v)K_1^-(u)R_{21}(u+v)K_2^-(v) = K_2^-(v)R_{12}(u+v)K_1^-(u)R_{21}(u-v), \\ R_{12}(v-u)K_1^+(u)\mathcal{R}_{21}(u+v)K_2^+(v) = K_2^+(v)\mathcal{R}_{12}(u+v)K_1^+(u)R_{21}(v-u), \end{cases}$$

where $\mathcal{R}(u) = R(-u-2\eta)$. Easy to check that

$$K^-(u) = \begin{pmatrix} \sinh(\xi^- + u) & 0 \\ 0 & \sinh(\xi^- - u) \end{pmatrix},$$

$$K^+(u) = \begin{pmatrix} \sinh(\xi^+ + u + \eta) & 0 \\ 0 & \sinh(\xi^+ - u - \eta) \end{pmatrix}$$

satisfy these equations for any $\xi^{\pm} \in \mathbb{C}$.

BQISM [Sklyanin 1988]

Define the **double row monodromy matrix** $\in \mathrm{End}\left(V_a \otimes V^{\otimes \mathcal{L}}\right)$:

$$T_a(u) = R_{a\mathcal{L}}(u-\varepsilon_{\mathcal{L}})...R_{a1}(u-\varepsilon_1)K_a^-(u)\times$$
$$\times R_{a1}(u+\varepsilon_1)...R_{a\mathcal{L}}(u+\varepsilon_{\mathcal{L}}) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}.$$

Trigonometric R-matrix satisfies: $R(u)R(-u) = I \otimes I$.

The **transfer matrix**

$$t(u) = tr_a\left(K_a^+(u)T_a(u)\right)$$

again satisfies

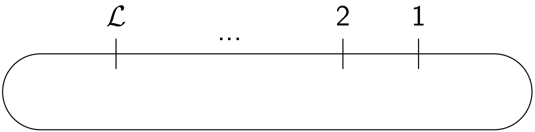
$$\boxed{[t(u), t(v)] = 0 \quad \forall u, v \in \mathbb{C}}$$

Thus, it is a generating function for the **integrals of motion**!

Periodic vs Boundary

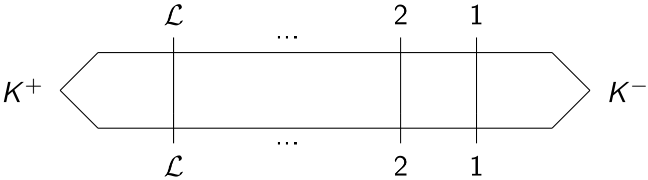
- **Periodic:**

$$t(u) = tr_a\left(R_{a\mathcal{L}}(u-\varepsilon_{\mathcal{L}})...R_{a2}(u-\varepsilon_2)R_{a1}(u-\varepsilon_1)\right).$$



- **Boundary:**

$$t(u) = tr_a\Big(K_a^+(u)R_{a\mathcal{L}}(u-\varepsilon_{\mathcal{L}})...R_{a1}(u-\varepsilon_1)\times$$
$$\times K_a^-(u)R_{a1}(u+\varepsilon_1)...R_{a\mathcal{L}}(u+\varepsilon_{\mathcal{L}})\Big).$$



Unified Approach

Modified **monodromy matrix** (shift $u \to u + \rho/2,\ \varepsilon_l \to \varepsilon_l + \rho/2$):

$$T_a(u) = R_{a\mathcal{L}}(u - \varepsilon_{\mathcal{L}})...R_{a1}(u - \varepsilon_1)K_a^-(u + \rho/2)\times$$
$$\times R_{a1}(u + \varepsilon_1 + \rho)...R_{a\mathcal{L}}(u + \varepsilon_{\mathcal{L}} + \rho).$$

The **transfer matrix**:

$$t(u) = tr_a\left(K_a^+(u + \rho/2)T_a(u)\right).$$

- $\rho \to 0$ yields Sklyanin's formulation,
- $\rho \to \infty$ yields periodic QISM:

$$\boxed{t(u) \xrightarrow{\rho\to\infty} tr_a\left(\begin{pmatrix} e^{-\eta\gamma} & 0 \\ 0 & e^{\eta\gamma} \end{pmatrix} R_{a\mathcal{L}}(u - \varepsilon_{\mathcal{L}})...R_{a1}(u - \varepsilon_1)\right)}$$

where

$$\gamma = \mathcal{L}/2 - N - \eta^{-1}(\xi^+ + \xi^-),\ \hat{N} = \sum_{j=1}^{\mathcal{L}} \hat{N}_j,\quad \hat{N}_j = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_j.$$

Algebraic Bethe Ansatz

Set $\rho = 0$ for the moment. Introduce $\tilde{a}(u) = \sinh(2u)a(u) - \sinh\eta d(u)$. Start with a reference state Ω and look for other eigenstates in the form

$$\Phi(v_1, ..., v_N) = C(v_1)...C(v_N)\Omega.$$

Eigenvalues:

$$\Lambda(u, v_1, ..., v_N) = \tilde{a}(u)\frac{\sinh(\xi^+ + u + \eta/2)}{\sinh 2u}\prod_{k=1}^{N}\frac{\sinh(u - v_k + \eta)\sinh(u + v_k + \eta)}{\sinh(u - v_k)\sinh(u + v_k)} +$$

$$+ d(u)\frac{\sinh(2u + \eta)\sinh(\xi^+ - u + \eta/2)}{\sinh 2u}\prod_{k=1}^{N}\frac{\sinh(u - v_k - \eta)\sinh(u + v_k - \eta)}{\sinh(u - v_k)\sinh(u + v_k)},$$

BAE:

$$\frac{\tilde{a}(v_k)}{d(v_k)\sinh(2v_k - \eta)}\frac{\sinh(\xi^+ + v_k + \eta/2)}{\sinh(\xi^+ - v_k + \eta/2)} = \prod_{i\neq k}^{N}\frac{\sinh(v_k - v_i - \eta)\sinh(v_k + v_i - \eta)}{\sinh(v_k - v_i + \eta)\sinh(v_k + v_i + \eta)}$$

Quasi-classical limit

Take $\Omega = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{\otimes \mathcal{L}}$ as a reference state and compute $\tilde{a}(u)$ and $d(u)$.

$$\frac{\tilde{a}(v_k)}{d(v_k)\sinh(2v_k - \eta)} = \frac{\sinh(\xi^- + v_k + \eta/2)}{\sinh(\xi^- - v_k + \eta/2)} \times \\ \times \prod_{l=1}^{\mathcal{L}} \frac{\sinh(v_k - \varepsilon_l - \eta/2)\sinh(v_k + \varepsilon_l - \eta/2)}{\sinh(v_k - \varepsilon_l + \eta/2)\sinh(v_k + \varepsilon_l + \eta/2)}.$$

If we substitute $\eta = 0$ the BAE will take the following form:

$$\frac{\sinh(\xi^- + v_k)\sinh(\xi^+ + v_k)}{\sinh(\xi^- - v_k)\sinh(\xi^+ - v_k)} = 1.$$

Choose $\xi^- = \xi^-(\eta),\ \xi^+ = \xi^+(\eta)$, so that this holds as $\eta \to 0$.

For example, take

$$\boxed{\xi^+ = \xi + \eta\alpha, \quad \xi^- = -\xi + \eta\beta}$$

Variable change 1

- **Trig. BQISM** (♣) (denote $\delta = -(\alpha + \beta + 1)$)

$$\delta\left(\coth(v_k - \xi) + \coth(v_k + \xi)\right) + \sum_{l=1}^{\mathcal{L}} \left(\coth(v_k - \varepsilon_l) + \coth(v_k + \varepsilon_l)\right) = \\ = 2\sum_{i \neq k}^{N} \left(\coth(v_k - v_i) + \coth(v_k + v_i)\right)$$

- **Trig. BQISM′** (♣′) can be obtained by $v_k \mapsto v_k + \rho/2, \ \varepsilon_l \mapsto \varepsilon_l + \rho/2$:

$$\delta\left(\coth(v_k + \rho/2 - \xi) + \coth(v_k + \rho/2 + \xi)\right) + \\ + \sum_{l=1}^{\mathcal{L}} \left(\coth(v_k - \varepsilon_l) + \coth(v_k + \varepsilon_l + \rho)\right) = \\ = 2\sum_{i \neq k}^{N} \left(\coth(v_k - v_i) + \coth(v_k + v_i + \rho)\right)$$

Attenuated limit

Note that $\coth(u) \to 1$ as $u \to \infty$.

Thus, as $\rho \to \infty$ **Trig. BQISM′** ($\clubsuit'$) tends to

$$2\delta + \sum_{l=1}^{\mathcal{L}} (\coth(v_k - \varepsilon_l) + 1) = 2\sum_{i \neq k}^{N} (\coth(v_k - v_i) + 1)\,.$$

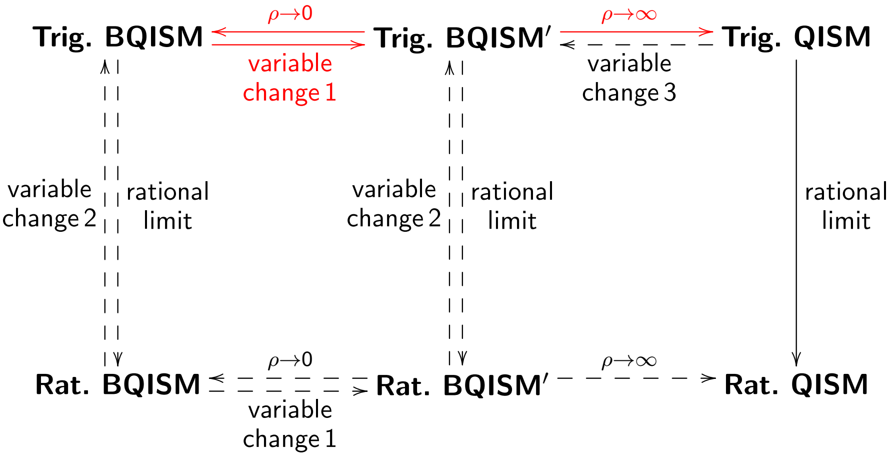
One can see that it is equivalent to **Trig. QISM** ($\diamondsuit$):

$$2(\delta + \mathcal{L}/2 - (N-1)) + \sum_{l=1}^{\mathcal{L}} \coth(v_k - \varepsilon_l) = 2\sum_{i \neq k}^{N} \coth(v_k - v_i).$$

Thus,

$$\boxed{\textbf{Trig. BQISM}' \ (\clubsuit') \xrightarrow{\rho \to \infty} \textbf{Trig. QISM} \ (\diamondsuit)}$$

Diagram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Rational limit

- **Rat. BQISM** (♠)

$$\boxed{\frac{\delta}{v_k^2-\xi^2}+\sum_{l=1}^{\mathcal{L}}\frac{1}{v_k^2-\varepsilon_l^2}=2\sum_{i\neq k}^{N}\frac{1}{v_k^2-v_i^2}}$$

Variable change

$$v_k \mapsto \sqrt{\exp(2v_k)+\xi^2},\ \ \varepsilon_l \mapsto \sqrt{\exp(2\varepsilon_l)+\xi^2}$$

yields **Trig. QISM** (♢). The inverse variable change is

$$v_k \mapsto \ln\sqrt{v_k^2-\xi^2},\ \ \varepsilon_l \mapsto \ln\sqrt{\varepsilon_l^2-\xi^2}.$$

Thus,

$$\boxed{\textbf{Rat. BQISM } (\spadesuit) \Leftrightarrow \textbf{Trig. QISM } (\diamondsuit)}$$

Attenuated limit

- **Rat. BQISM′** (♠′) (obtained by $v_k \mapsto v_k + \rho/2,\ \varepsilon_l \mapsto \varepsilon_l + \rho/2$)

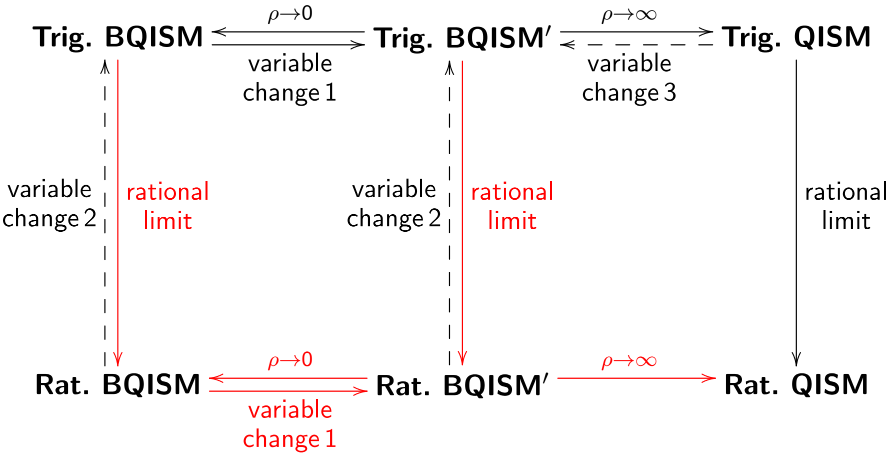
$$\boxed{\begin{aligned}&\frac{\delta}{(v_k+\rho/2)^2-\xi^2}+\sum_{l=1}^{\mathcal{L}}\frac{1}{(v_k+\rho/2)^2-(\varepsilon_l+\rho/2)^2}=\\&=2\sum_{j\neq k}^{N}\frac{1}{(v_k+\rho/2)^2-(v_j+\rho/2)^2}\end{aligned}}$$

$$\delta+\sum_{l=1}^{\mathcal{L}}\frac{v_k^2+\rho v_k+\rho^2/4-\xi^2}{v_k^2-\varepsilon_l^2+\rho(v_k-\varepsilon_l)}=2\sum_{i\neq k}^{N}\frac{v_k^2+\rho v_k+\rho^2/4-\xi^2}{v_k^2-v_i^2+\rho(v_k-v_i)}.$$

Rescale $\delta=\rho\gamma/2$ and consider $\rho\to\infty$. We obtain **Rat. QISM** (♡):

$$2\gamma+\sum_{l=1}^{\mathcal{L}}\frac{1}{v_k-\varepsilon_l}=2\sum_{j\neq k}^{N}\frac{1}{v_k-v_j}.$$

Diagram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Rat. QISM

Variable change 2

Substitute $\boxed{v_k \mapsto \frac{y_k - y_k^{-1}}{2},\ \varepsilon_l \mapsto \frac{z_l - z_l^{-1}}{2},\ \xi \mapsto \frac{\chi - \chi^{-1}}{2}}$ into (♠):

$$\delta \frac{y_k^2 + y_k^{-2}}{y_k^2 + y_k^{-2} - \chi^2 - \chi^{-2}} + \sum_{l=1}^{\mathcal{L}} \frac{y_k^2 - y_k^{-2}}{y_k^2 + y_k^{-2} - z_l^2 - z_l^{-2}} = 2\sum_{i \neq k}^{N} \frac{y_k^2 - y_k^{-2}}{y_k^2 + y_k^{-2} - y_i^2 - y_i^{-2}}.$$

Using

$$\frac{y_k^2 - y_k^{-2}}{y_k^2 + y_k^{-2} - z_l^2 - z_l^{-2}} = \frac{y_k^2}{y_k^2 - z_l^2} + \frac{1}{y_k^2 z_l^2 - 1}$$

we obtain

$$\delta\left(\frac{y_k^2}{y_k^2 - \chi^2} + \frac{1}{y_k^2\chi^2 - 1}\right) + \sum_{l=1}^{\mathcal{L}}\left(\frac{y_k^2}{y_k^2 - z_l^2} + \frac{1}{y_k^2 z_l^2 - 1}\right) =$$

$$= \sum_{i \neq k}^{N}\left(\frac{2y_k^2}{y_k^2 - y_i^2} + \frac{2}{y_k^2 y_i^2 - 1}\right).$$

Variable change 2

Rewrite

$$\delta\left(\frac{y_k^2+\chi^2}{y_k^2-\chi^2}+\frac{y_k^2\chi^2+1}{y_k^2\chi^2-1}\right)+\sum_{l=1}^{\mathcal{L}}\left(\frac{y_k^2+z_l^2}{y_k^2-z_l^2}+\frac{y_k^2z_l^2+1}{y_k^2z_l^2-1}\right)=$$

$$=2\sum_{i\neq k}^{N}\left(\frac{y_k^2+y_i^2}{y_k^2-y_i^2}+\frac{y_k^2y_i^2+1}{y_k^2y_i^2-1}\right)$$

Now make a variable change

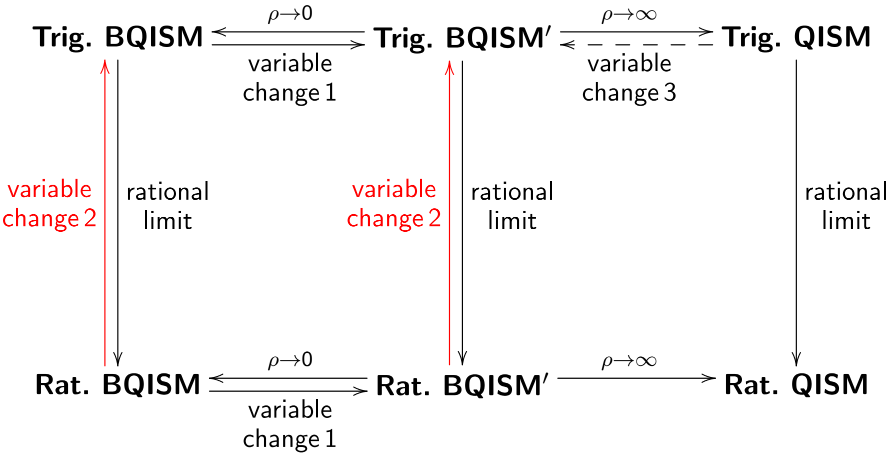
$$y_k\mapsto\exp v_k,\;\; z_l\mapsto\exp\varepsilon_l,\;\; \chi\mapsto\exp\xi$$

to turn it into **Trig. BQISM (♣)**. **Variable change 2** is the composition:

$$v_k\mapsto\frac{\exp v_k-\exp(-v_k)}{2}=\sinh v_k,\;\; \varepsilon_l\mapsto\sinh\varepsilon_l,\;\; \xi\mapsto\sinh\xi.$$

Rat. BQISM (♠) $\xrightarrow{\text{variable change 2}}$ **Trig. BQISM (♣)**

Diagram



$\rho \to 0$
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variable change 1
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rational limit
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$\rho \to 0$
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Rat. BQISM
Rat. BQISM′
Rat. QISM
variable change 1

Variable change 3

Trig. QISM ($\diamondsuit$) turns into **Rat. BQISM** ($\spadesuit$) by a **variable change 4**

$$v_k \mapsto \ln\sqrt{v_k^2 - \xi^2}, \ \varepsilon_l \mapsto \ln\sqrt{\varepsilon_l^2 - \xi^2}.$$

Variable change 3 is a composition: **Trig. QISM** ($\diamondsuit$)

$$\xrightarrow{\text{variable change 4}} \textbf{Rat. BQISM } (\spadesuit) \xrightarrow{\text{variable change 2}}$$

$$\textbf{Trig. BQISM } (\clubsuit) \xrightarrow{\text{variable change 1}} \textbf{Trig. BQISM}' \ (\clubsuit'),$$

which gives

$$v_k \mapsto \ln\sqrt{\sinh^2(v_k + \rho/2) - \sinh^2\xi},$$
$$\varepsilon_l \mapsto \ln\sqrt{\sinh^2(\varepsilon_l + \rho/2) - \sinh^2\xi}.$$

Thank you for your attention!