An integrable case of the $p + ip$ pairing Hamiltonian interacting with its environment

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The pairing model interacting with its environment

Let $c_{\mathbf{k}}, c_{\mathbf{k}}^{\dagger}$ denote the annihilation and creation operators, $\mathbf{k} = (k_x, k_y)$:

$$\{c_{\mathbf{k}}, c_{\mathbf{k}'}\} = \{c_{\mathbf{k}}^{\dagger}, c_{\mathbf{k}'}^{\dagger}\} = 0, \ \{c_{\mathbf{k}}, c_{\mathbf{k}'}^{\dagger}\} = \delta_{\mathbf{k}\mathbf{k}'} I.$$

- The **isolated** $p + ip$ pairing Hamiltonian:

$$H_0 = \sum_{\mathbf{k}} \frac{|\mathbf{k}|^2}{2m} c_{\mathbf{k}}^{\dagger} c_{\mathbf{k}} - \frac{G}{4m} \sum_{\mathbf{k} \neq \pm \mathbf{k}'} (k_x + ik_y)(k_x' - ik_y') c_{\mathbf{k}}^{\dagger} c_{-\mathbf{k}}^{\dagger} c_{-\mathbf{k}'} c_{\mathbf{k}'}.$$

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- Consider this Hamiltonian with an **extra term**:

$$H = H_0 + \frac{\Gamma}{2} \sum_{\mathbf{k}} \Big((k_x + ik_y) c_{\mathbf{k}}^\dagger c_{-\mathbf{k}}^\dagger + (k_x - ik_y) c_{-\mathbf{k}} c_{\mathbf{k}} \Big).$$

The extra term can be interpreted as creation and annihilation of pairs of fermions, resulting from **interaction with the environment**.

Reformulation in terms of spin operators

Set $z_{\mathbf{k}} = |\mathbf{k}|$, $k_x + ik_y = |\mathbf{k}|\exp(i\phi_{\mathbf{k}})$. Introduce the **spin operators**

$$S_{\mathbf{k}}^{+} = \exp(i\phi_{\mathbf{k}})c_{\mathbf{k}}^{\dagger}c_{-\mathbf{k}}^{\dagger}, \; S_{\mathbf{k}}^{-} = \exp(-i\phi_{\mathbf{k}})c_{-\mathbf{k}}c_{\mathbf{k}}, \; S_{\mathbf{k}}^{z} = c_{\mathbf{k}}^{\dagger}c_{-\mathbf{k}}^{\dagger}c_{-\mathbf{k}}c_{\mathbf{k}} - \frac{I}{2}.$$

They satisfy $\mathfrak{su}(2)$ **commutation relations**

$$[S_{\mathbf{k}}^{z}, S_{\mathbf{k}}^{\pm}] = \pm S_{\mathbf{k}}^{\pm}, \; [S_{\mathbf{k}}^{+}, S_{\mathbf{k}}^{-}] = 2S_{\mathbf{k}}^{z}.$$

Restricting to paired states and using integers $k = 1, \dots, \mathcal{L}$ to enumerate the pairs $(\mathbf{k}, -\mathbf{k})$ we can rewrite $(m = 1)$:

$$H_0 = \sum_{k=1}^{\mathcal{L}} z_k^2 S_k^z - G \sum_{k=1}^{\mathcal{L}} \sum_{j \neq k} z_k z_j S_k^{+} S_j^{-}$$

and

$$\boxed{H = H_0 + \Gamma \sum_{k=1}^{\mathcal{L}} z_k \left(S_k^{+} + S_k^{-}\right)}$$

Summary and outline

Isolated pairing model (H_0) [Ibañez, Links, Sierra, Zhao 2009]

- integrable by the **Quantum Inverse Scattering Method (QISM)** using the **trigonometric solution** of the Yang-Baxter Equation (YBE),
- exhibits $\mathfrak{u}(1)$-symmetry: $[H_0, S^z] = 0$, where $S^z = \sum_{k=1}^{\mathcal{L}} S_k^z$,
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This talk: model interacting with its environment (H)

- integrable by the **Boundary QISM (BQISM)** using the **rational solution** of the YBE and one of the K-matrices being non-diagonal,
- no longer exhibits $\mathfrak{u}(1)$-symmetry $\Rightarrow$ ABA is not obviously applicable,
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Key ingredients

- The Hilbert space of states

$$\mathcal{H} = \bigotimes_{j=1}^{\mathcal{L}} V_j = V^{\otimes \mathcal{L}}, \text{ where } V = \mathbb{C}^2 \text{ spin-}1/2 \text{ rep. space of } \mathfrak{su}(2).$$

- The **rational R-matrix** $(\eta \in \mathbb{C}, \quad P(u \otimes v) = v \otimes u, \ \forall u, v \in V)$

$$R(u) = uI \otimes I + \eta P = \begin{pmatrix} u+\eta & 0 & 0 & 0 \\ 0 & u & \eta & 0 \\ 0 & \eta & u & 0 \\ 0 & 0 & 0 & u+\eta \end{pmatrix} \in \mathrm{End}(V \otimes V)$$

satisfies $R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v)$.

- The **Lax operator** ($V_a = V$ is the auxiliary space)

$$L_{aj}(u) = I + \frac{\eta}{u}\begin{pmatrix} S_j^z & S_j^- \\ S_j^+ & -S_j^z \end{pmatrix} = \frac{1}{u} R_{aj}(u - \eta/2) \in \mathrm{End}(V_a \otimes V_j)$$

satisfies $R_{ab}(u-v)L_{aj}(u)L_{bj}(v) = L_{bj}(v)L_{aj}(u)R_{ab}(u-v)$.

BQISM [Sklyanin 1988]

- **Reflection equations** ($K^{\pm}(u) \in \mathrm{End}(V)$, $\mathcal{R}(u) = R(-u-2\eta)$):

$$\begin{cases} R_{12}(u-v)K_1^-(u)R_{21}(u+v)K_2^-(v) = K_2^-(v)R_{12}(u+v)K_1^-(u)R_{21}(u-v), \\ R_{12}(v-u)K_1^+(u)\mathcal{R}_{21}(u+v)K_2^+(v) = K_2^+(v)\mathcal{R}_{12}(u+v)K_1^+(u)R_{21}(v-u), \end{cases}$$

- Consider the following K-**matrices** ($\xi^{\pm}, \phi, \psi \in \mathbb{C}$):

$$K^-(u) = \begin{pmatrix} \xi^- + u - \eta/2 & 0 \\ 0 & \xi^- - u + \eta/2 \end{pmatrix},$$

$$K^+(u) = \begin{pmatrix} \xi^+ + u + \eta/2 & \psi(u+\eta/2) \\ \phi(u+\eta/2) & \xi^+ - u - \eta/2 \end{pmatrix}.$$

- The **transfer matrix** $\in \mathrm{End}(\mathcal{H})$

$$t(u) = \mathrm{tr}_a\left(K_a^+(u)L_{a\mathcal{L}}(u-\varepsilon_{\mathcal{L}})...L_{a1}(u-\varepsilon_1)K_a^-(u)L_{a1}(u+\varepsilon_1)...L_{a\mathcal{L}}(u+\varepsilon_{\mathcal{L}})\right)$$

satisfies $[t(u), t(v)] = 0 \quad \forall u, v \in \mathbb{C} \quad \Rightarrow$ can be used as a generating function for the conserved operators.

Constructing the conserved operators

Take the **quasi-classical limit** to construct the conserved operators:

$$\lim_{u \to \varepsilon_j} (u - \varepsilon_j) t(u) = \eta^2 \tau_j + o(\eta^2).$$

Condition: for it to be well-defined the K-matrices have to satisfy

$$K^+(u) K^-(u) \to f(u) I \quad \text{as} \quad \eta \to 0. \qquad (\dagger)$$

Assume that parameters depend on η as follows:

$$\boxed{\xi^+ = \xi + \eta\alpha, \quad \xi^- = -\xi + \eta\beta, \quad \psi = \eta\gamma, \quad \phi = \eta\lambda}$$

Then $(\dagger)$ is satisfied and the **conserved operators** are

$$\tau_j^* = \sum_{k \neq j}^{\mathcal{L}} \frac{4\varepsilon_j^2}{\varepsilon_j^2 - \varepsilon_k^2} S_j^z S_k^z + \sum_{k \neq j}^{\mathcal{L}} \frac{2\varepsilon_j \varepsilon_k}{\varepsilon_j^2 - \varepsilon_k^2} (S_j^+ S_k^- + S_j^- S_k^+) +$$
$$+ 2(\alpha + \beta) S_j^z + \gamma \varepsilon_j S_j^+ - \lambda \varepsilon_j S_j^-.$$

Constructing the Hamiltonian

$$\sum_{j=1}^{\mathcal{L}} \varepsilon_j^{-2}\tau_j^* = -2\sum_{j,k:j<k} \varepsilon_j^{-1}\varepsilon_k^{-1}(S_j^+S_k^- + S_j^-S_k^+) + 2(\alpha+\beta)\sum_{j=1}^{\mathcal{L}} \varepsilon_j^{-2}S_j^z + \\ + \gamma\sum_{j=1}^{\mathcal{L}} \varepsilon_j^{-1}S_j^+ - \lambda\sum_{j=1}^{\mathcal{L}} \varepsilon_j^{-1}S_j^- \equiv H'.$$

Making the change of variable $z_j = \varepsilon_j^{-1}$ we obtain

$$H' = 2(\alpha+\beta)\sum_{j=1}^{\mathcal{L}} z_j^2 S_j^z - 2\sum_{j,k:j<k} z_j z_k (S_j^+S_k^- + S_j^-S_k^+) + \gamma\sum_{j=1}^{\mathcal{L}} z_j S_j^+ - \lambda\sum_{j=1}^{\mathcal{L}} z_j S_j^-.$$

Set $\gamma = -\lambda$. Then $H = \frac{1}{2}GH'$ with $\alpha+\beta = G^{-1}$ and $\gamma = 2\Gamma G^{-1}$:

$$\boxed{H = \sum_{k=1}^{\mathcal{L}} z_k^2 S_k^z - G\sum_{k=1}^{\mathcal{L}}\sum_{j\neq k} z_k z_j S_k^+ S_j^- + \Gamma\sum_{k=1}^{\mathcal{L}} z_k\left(S_k^+ + S_k^-\right)}$$

The energy spectrum

[Cao, Yang, Shi, Wang 2013]: **Off-Diagonal Bethe Ansatz (ODBA)** (a method of solution for models where $\mathfrak{u}(1)$ symmetry is broken).
[Hao, Cao, Yang, Yang 2015]: ODBA applied to the XXX Gaudin model.
Utilising this result we obtain

- the **eigenvalues** of the Hamiltonian H (the **energy spectrum**)

$$\boxed{E = (1+G)\sum_{i=1}^{\mathcal{L}} y_i - \frac{1}{2}\sum_{j=1}^{\mathcal{L}} z_j^2 + \Gamma^2 G^{-1} \sum_{i=1}^{\mathcal{L}} \frac{\prod_{j=1}^{\mathcal{L}}(1 - y_i z_j^{-2})}{\prod_{k\neq i}^{\mathcal{L}}(1 - y_i y_k^{-1})}}$$

- subject to the **Bethe Ansatz Equations** $(k = 1, \dots, \mathcal{L})$

$$\boxed{1 + G^{-1} + \sum_{i\neq k}^{\mathcal{L}} \frac{2y_i}{y_i - y_k} - \sum_{l=1}^{\mathcal{L}} \frac{z_l^2}{y_k - z_l^2} = -\Gamma^2 G^{-2} \frac{1}{y_k} \frac{\prod_{l=1}^{\mathcal{L}}(1 - y_k z_l^{-2})}{\prod_{i\neq k}^{\mathcal{L}}(1 - y_k y_i^{-1})}}$$

Thank you for your attention!

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