Properties of the Bethe Ansatz equations for Richardson-Gaudin models

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Introduction

- **Heisenberg picture** of quantum mechanics: $\frac{d}{dt}A = \frac{i}{\hbar}[H, A]$, where $[H, A] = HA - AH$ and H is the Hamiltonian.
- **Conserved operator** is an operator that commutes with the Hamiltonian of the system: $[C, H] = 0$.
- **Quantum integrable system** (naive definition): a system that admits a "complete" set of mutually commuting conserved operators ($\Rightarrow$ simultaneously diagonalisable).
- **Exactly solvable system**: the eigenvalues and the eigenstates of the Hamiltonian can be exactly determined.

QISM

- **Quantum Inverse Scattering Method (QISM)**: a method for construction and solution of quantum integrable systems.

QISM: ingredients $\Rightarrow \ldots \Rightarrow \{C_j\}$ mutually commuting operators

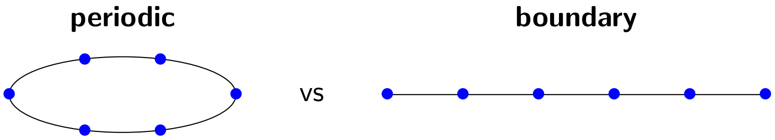
- Define the Hamiltonian $H = f(\{C_j\}) \Rightarrow \{C_j\}$ are mutually commuting conserved operators $\Rightarrow$ quantum integrability.

- **Algebraic Bethe Ansatz** is incorporated into the QISM to exactly determine the eigenstates and the eigenvalues of $\{C_j\}$.

- These eigenstates and eigenvalues depend on a set of parameters satisfying the **Bethe Ansatz Equations**.

Some history and setting up the problem

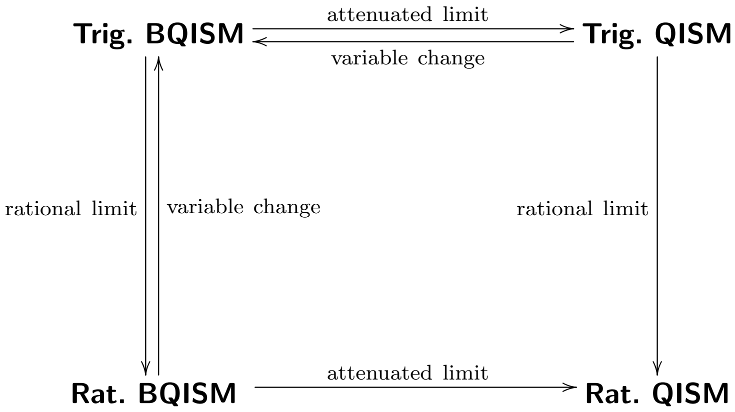
- **QISM** for twisted periodic boundary conditions [Faddeev, Kulish, Sklyanin, Takhtadzhan 1979].
- **Boundary QISM (BQISM)** for open-boundary conditions, for the case of the Heisenberg XXZ spin chain [Sklyanin 1988].



- **Richardson-Gaudin models**: quasi-classical limit of the (B)QISM.
- **Question:** what is the effect of the "boundary" for the Richardson-Gaudin models?
  - Gaudin magnet with boundary [Hikami 1995].
  - Quasi-classical limit of the BQISM [Di Lorenzo et al. 2002].
  - Generalized Gaudin systems [Skrypnyk 2006, 2007, 2010].
  - Trigonometric Gaudin model [Cirilio António et al. 2013].

Outline

For the **Richardson-Gaudin models**:



Twisted periodic case

Algebraic Bethe Ansatz: start with the **reference state** Ω and construct other eigenstates of $\{C_j\}$ in the form

$$\Phi(v_1, ..., v_N) = C(v_1)...C(v_N)\Omega,$$

where $C(u)$ are certain operators depending on a complex parameter.

Parameters $v_k \in \mathbb{C}$ must satisfy the **Bethe Ansatz Equations (BAE)**:

$$e^{-2\eta\gamma} \prod_{l=1}^{\mathcal{L}} \frac{\sinh(v_k - \varepsilon_l - \eta/2)}{\sinh(v_k - \varepsilon_l + \eta/2)} = \prod_{i \neq k}^{N} \frac{\sinh(v_k - v_i - \eta)}{\sinh(v_k - v_i + \eta)}, \quad k = 1, ..., N$$

Take first non-zero term as $\eta \to 0$ to obtain the Richardson-Gaudin models in the **quasi-classical limit**.

Richardson-Gaudin models

- **Trig. QISM** [Amico et al. 2001], [Dukelsky et al. 2001]

$$\boxed{2\gamma + \sum_{l=1}^{\mathcal{L}} \coth(v_k - \varepsilon_l) = 2\sum_{i\neq k}^{N} \coth(v_k - v_i)} \qquad (1)$$

- **Rat. QISM** [Richardson 1963]

(via the **rational limit**: $\nu \coth(\nu x) \to \frac{1}{x}$ as $\nu \to 0$)

$$\boxed{2\gamma' + \sum_{l=1}^{\mathcal{L}} \frac{1}{v_k - \varepsilon_l} = \sum_{i\neq k}^{N} \frac{2}{v_k - v_i}} \qquad (0)$$

Boundary case

The BAE from Sklyanin's BQISM:

$$\boxed{\begin{aligned}&\frac{\sinh(\xi^+ + v_k + \eta/2)}{\sinh(\xi^+ - v_k + \eta/2)}\frac{\sinh(\xi^- + v_k + \eta/2)}{\sinh(\xi^- - v_k + \eta/2)}\times\\ &\times\prod_{l=1}^{\mathcal{L}}\frac{\sinh(v_k - \varepsilon_l - \eta/2)\sinh(v_k + \varepsilon_l - \eta/2)}{\sinh(v_k - \varepsilon_l + \eta/2)\sinh(v_k + \varepsilon_l + \eta/2)} =\\ &= \prod_{i\neq k}^{N}\frac{\sinh(v_k - v_i - \eta)\sinh(v_k + v_i - \eta)}{\sinh(v_k - v_i + \eta)\sinh(v_k + v_i + \eta)}\end{aligned}}$$

If we substitute $\eta = 0$ the BAE will take the following form:

$$\frac{\sinh(\xi^+ + v_k)\sinh(\xi^- + v_k)}{\sinh(\xi^+ - v_k)\sinh(\xi^- - v_k)} = 1. \qquad (\dagger)$$

Assume $\xi^+ = \xi^+(\eta),\ \xi^- = \xi^-(\eta)$, so that $(\dagger)$ holds as $\eta \to 0$:

$$\boxed{\xi^+ = \xi + \eta\alpha, \quad \xi^- = -\xi + \eta\beta}$$

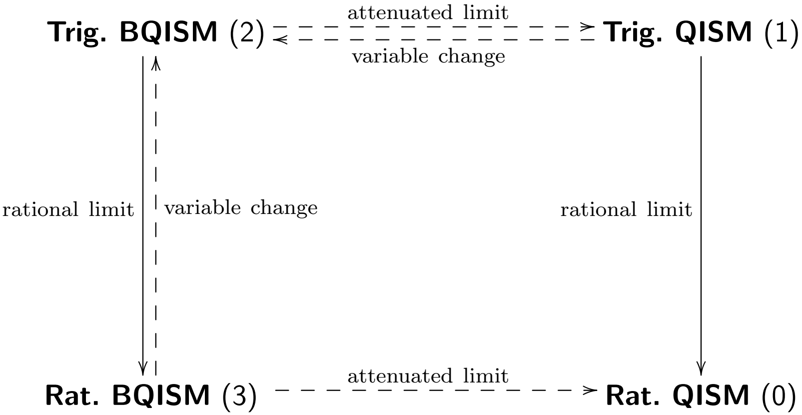
Richardson-Gaudin models with boundary

- **Trig. BQISM** (denote $\delta = -(\alpha + \beta + 1)$) (2)

$$\delta\left(\coth(v_k - \xi) + \coth(v_k + \xi)\right) + \sum_{l=1}^{\mathcal{L}} \left(\coth(v_k - \varepsilon_l) + \coth(v_k + \varepsilon_l)\right) = = 2\sum_{i \neq k}^{N} \left(\coth(v_k - v_i) + \coth(v_k + v_i)\right)$$

- **Rat. BQISM** (by taking the rational limit)

$$\delta\left(\frac{1}{v_k - \xi} + \frac{1}{v_k + \xi}\right) + \sum_{l=1}^{\mathcal{L}} \left(\frac{1}{v_k - \varepsilon_l} + \frac{1}{v_k + \varepsilon_l}\right) = = 2\sum_{i \neq k}^{N} \left(\frac{1}{v_k - v_i} + \frac{1}{v_k + v_i}\right) \tag{3}$$


Trig. BQISM (2)
attenuated limit
variable change
Trig. QISM (1)
rational limit
variable change
rational limit
Rat. BQISM (3)
attenuated limit
Rat. QISM (0)

Attenuated limit

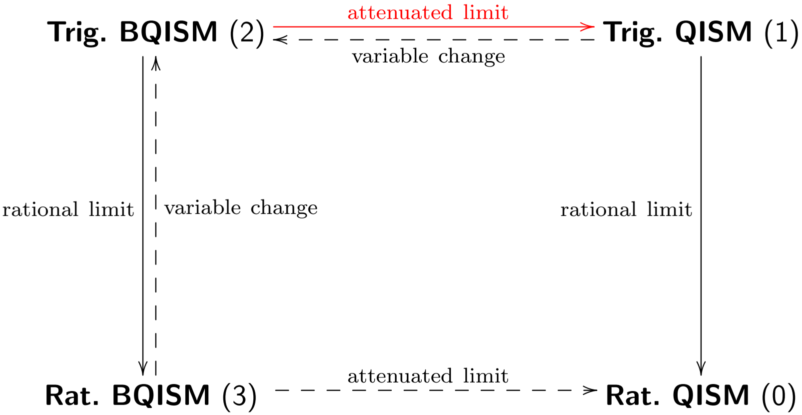
Make a variable change $v_k \mapsto v_k + \rho/2$, $\varepsilon_l \mapsto \varepsilon_l + \rho/2$ in (2):

$$\boxed{\begin{aligned} &\delta\left(\coth(v_k + \rho/2 - \xi) + \coth(v_k + \rho/2 + \xi)\right) + \\ &+ \sum_{l=1}^{\mathcal{L}} \left(\coth(v_k - \varepsilon_l) + \coth(v_k + \varepsilon_l + \rho)\right) = \\ &= 2 \sum_{i \neq k}^{N} \left(\coth(v_k - v_i) + \coth(v_k + v_i + \rho)\right) \end{aligned}} \tag{2'}$$

Note that $\coth x \to 1$ as $x \to \infty$. Thus, as $\rho \to \infty$, (2′) tends to

$$2\delta + \sum_{l=1}^{\mathcal{L}} \left(\coth(v_k - \varepsilon_l) + 1\right) = 2 \sum_{i \neq k}^{N} \left(\coth(v_k - v_i) + 1\right).$$

One can see that it is equivalent to **Trig. QISM** (1).


attenuated limit
Trig. BQISM (2)
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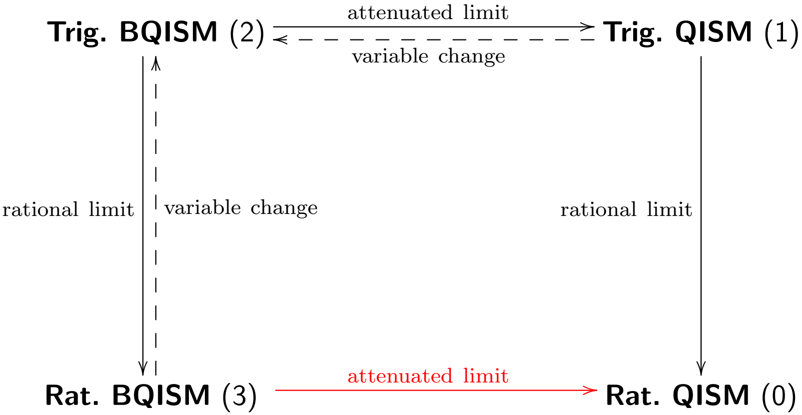
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Rational limit

Rat. BQISM (3) can be rewritten as follows:

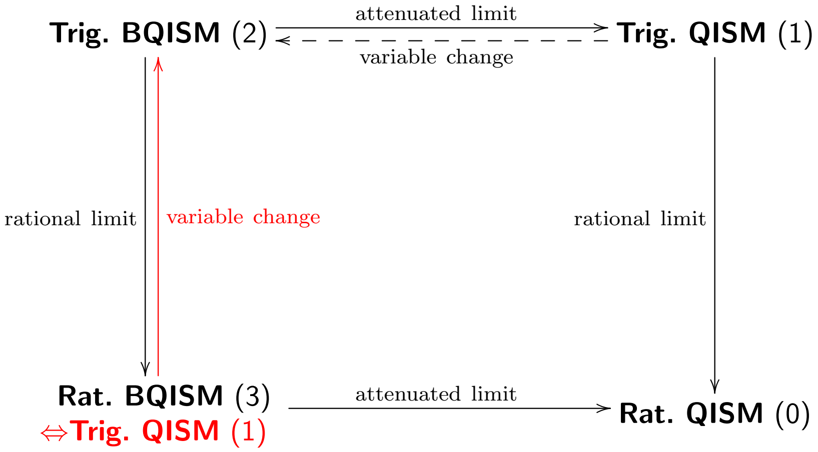
$$\boxed{\frac{\delta}{v_k^2-\xi^2}+\sum_{l=1}^{\mathcal{L}}\frac{1}{v_k^2-\varepsilon_l^2}=2\sum_{i\neq k}^{N}\frac{1}{v_k^2-v_i^2}} \tag{3}$$

One can show that $\boxed{\textbf{Rat. BQISM } (3) \iff \textbf{Trig. QISM } (1)}$ via an invertible variable change:

$$v_k \mapsto \sqrt{\exp(2v_k)+\xi^2},\ \ \varepsilon_l \mapsto \sqrt{\exp(2\varepsilon_l)+\xi^2}.$$

Furthermore, $\boxed{\textbf{Rat. BQISM } (3) \longrightarrow \textbf{Trig. BQISM } (2)}$ via another variable change:

$$v_k \mapsto \sinh v_k,\ \ \varepsilon_l \mapsto \sinh\varepsilon_l,\ \ \xi \mapsto \sinh\xi.$$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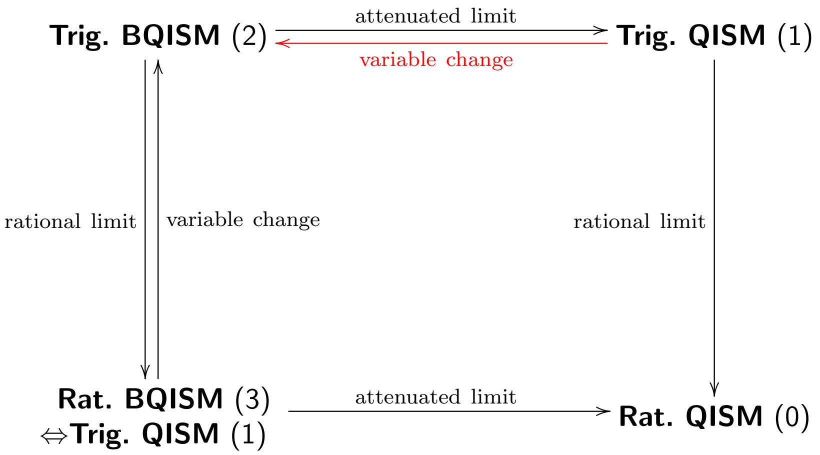

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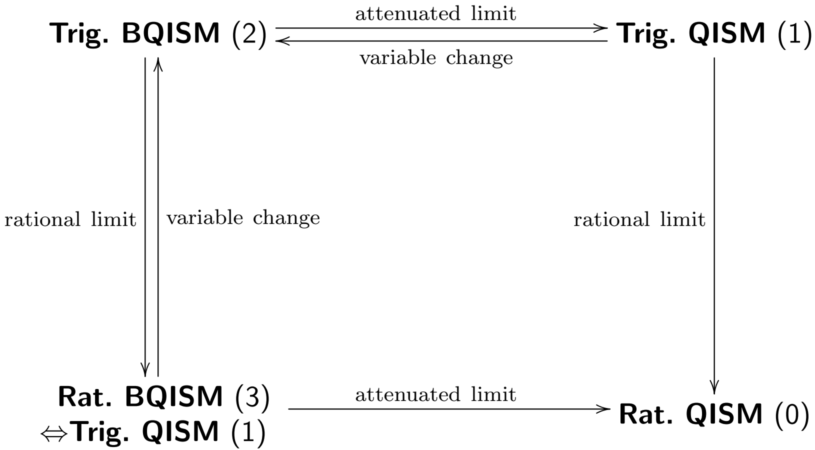
Thank you for your attention!



*On the boundaries of quantum integrability for the spin-*1/2 *Richardson–Gaudin system*, Nucl. Phys. B **886**, 364-398 (2014)