An integrable case of the $p + ip$ pairing Hamiltonian interacting with its environment

arXiv:1507.04068

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Quantum Integrable Models

- **Quantum integrable model** (naive definition): a model that admits a "complete" set of mutually commuting conserved operators.
- **Exactly solved model**: the eigenvalues and the eigenstates of the Hamiltonian can be exactly determined.
- **Quantum Inverse Scattering Method (QISM)**[1]: a method for construction and solution of quantum integrable models.

QISM: ingredients $\Rightarrow \ldots \Rightarrow \{C_j\}$ mutually commuting operators

- Construct the Hamiltonian as $H = f(\{C_j\}) \Rightarrow \{C_j\}$ are mutually commuting conserved operators $\Rightarrow$ quantum integrability.
- **Algebraic Bethe Ansatz**[2] is incorporated into the QISM to exactly determine the eigenstates and the eigenvalues of $\{C_j\}$.

[1] Faddeev et al 1979

[2] Bethe 1931, Faddeev et al 1979

The pairing model interacting with its environment

Let $c_{\mathbf{k}}, c_{\mathbf{k}}^{\dagger}$ denote the annihilation and creation operators, $\mathbf{k} = (k_x, k_y)$:

$$\{c_{\mathbf{k}}, c_{\mathbf{k}'}\} = \{c_{\mathbf{k}}^{\dagger}, c_{\mathbf{k}'}^{\dagger}\} = 0, \ \{c_{\mathbf{k}}, c_{\mathbf{k}'}^{\dagger}\} = \delta_{\mathbf{k}\mathbf{k}'} I.$$

- The **isolated** $p + ip$ pairing Hamiltonian:

$$H_0 = \sum_{\mathbf{k}} \frac{|\mathbf{k}|^2}{2m} c_{\mathbf{k}}^{\dagger} c_{\mathbf{k}} - \frac{G}{4m} \sum_{\mathbf{k} \neq \pm \mathbf{k}'} (k_x + ik_y)(k'_x - ik'_y) c_{\mathbf{k}}^{\dagger} c_{-\mathbf{k}}^{\dagger} c_{-\mathbf{k}'} c_{\mathbf{k}'}.$$

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- Consider this Hamiltonian with an **extra term**:

$$H = H_0 + \frac{\Gamma}{2} \sum_{\mathbf{k}} \Big((k_x + ik_y) c_{\mathbf{k}}^{\dagger} c_{-\mathbf{k}}^{\dagger} + (k_x - ik_y) c_{-\mathbf{k}} c_{\mathbf{k}} \Big).$$

The extra term can be interpreted as creation and annihilation of pairs of fermions, resulting from **interaction with the environment**.

Reformulation in terms of spin operators

Set $z_{\mathbf{k}} = |\mathbf{k}|$, $k_x + ik_y = |\mathbf{k}|\exp(i\phi_{\mathbf{k}})$. Introduce the **spin operators**

$$S_{\mathbf{k}}^{+} = \exp(i\phi_{\mathbf{k}})c_{\mathbf{k}}^{\dagger}c_{-\mathbf{k}}^{\dagger},\ S_{\mathbf{k}}^{-} = \exp(-i\phi_{\mathbf{k}})c_{-\mathbf{k}}c_{\mathbf{k}},\ S_{\mathbf{k}}^{z} = c_{\mathbf{k}}^{\dagger}c_{-\mathbf{k}}^{\dagger}c_{-\mathbf{k}}c_{\mathbf{k}} - \frac{I}{2}.$$

They satisfy $\mathfrak{su}(2)$ **commutation relations**

$$[S_{\mathbf{k}}^{z}, S_{\mathbf{k}}^{\pm}] = \pm S_{\mathbf{k}}^{\pm},\ [S_{\mathbf{k}}^{+}, S_{\mathbf{k}}^{-}] = 2S_{\mathbf{k}}^{z}.$$

Restricting to paired states and using integers $k = 1, \ldots, \mathcal{L}$ to enumerate the pairs $(\mathbf{k}, -\mathbf{k})$ we can rewrite $(m = 1)$:

$$H_0 = \sum_{k=1}^{\mathcal{L}} z_k^2 S_k^z - G \sum_{k=1}^{\mathcal{L}} \sum_{j \neq k} z_k z_j S_k^+ S_j^-$$

and

$$\boxed{H = H_0 + \Gamma \sum_{k=1}^{\mathcal{L}} z_k \left(S_k^+ + S_k^-\right)}$$

Summary and outline

Isolated pairing model[3] (H_0)

- integrable by the **Quantum Inverse Scattering Method (QISM)** using the **trigonometric solution** of the Yang-Baxter Equation (YBE),
- solved by the **Algebraic Bethe Ansatz (ABA)**,
- exhibits $\mathfrak{u}(1)$-symmetry: $[H_0, S^z] = 0$, where $S^z = \sum_{k=1}^{\mathcal{L}} S_k^z$.

[3] Ibañez, Links, Sierra, Zhao 2009

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Isolated pairing model[3] (H_0)

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This talk: model interacting with its environment (H)

- integrable by the **Boundary QISM (BQISM)**[4] using the **rational solution** of the YBE and one of the K-matrices being non-diagonal,
- no longer exhibits $\mathfrak{u}(1)$-symmetry $\Rightarrow$ ABA is not obviously applicable,
- solved using the **Off-Diagonal Bethe Ansatz (ODBA)**[5].

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[3] Ibañez, Links, Sierra, Zhao 2009

[4] Sklyanin 1988

[5] Cao, Yang, Shi, Wang 2013

Key ingredients of the (B)QISM

- The Hilbert space of states

$$\mathcal{H} = \bigotimes_{j=1}^{\mathcal{L}} V_j = V^{\otimes \mathcal{L}}, \text{ where } V = \mathbb{C}^2 \text{ spin-1/2 rep. space of } \mathfrak{su}(2).$$

- The **rational** R**-matrix** ($\eta \in \mathbb{C}$, $P(\vec{x} \otimes \vec{y}) = \vec{y} \otimes \vec{x}$, $\forall \vec{x}, \vec{y} \in V$)

$$R(u) = uI \otimes I + \eta P = \begin{pmatrix} u+\eta & 0 & 0 & 0 \\ 0 & u & \eta & 0 \\ 0 & \eta & u & 0 \\ 0 & 0 & 0 & u+\eta \end{pmatrix} \in \mathrm{End}(V \otimes V)$$

satisfies $R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v)$.

- The **Lax operator** ($V_a = V$ is the auxiliary space)

$$L_{aj}(u) = I + \frac{\eta}{u}\begin{pmatrix} S_j^z & S_j^- \\ S_j^+ & -S_j^z \end{pmatrix} = \frac{1}{u} R_{aj}(u - \eta/2) \in \mathrm{End}(V_a \otimes V_j)$$

satisfies $R_{ab}(u-v)L_{aj}(u)L_{bj}(v) = L_{bj}(v)L_{aj}(u)R_{ab}(u-v)$.

Sklyanin's BQISM

- **Reflection equations** ($K^{\pm}(u) \in \mathrm{End}(V)$, $\mathcal{R}(u) = R(-u-2\eta)$):

$$\begin{cases} R_{12}(u-v)K_1^-(u)R_{21}(u+v)K_2^-(v) = K_2^-(v)R_{12}(u+v)K_1^-(u)R_{21}(u-v), \\ R_{12}(v-u)K_1^+(u)\mathcal{R}_{21}(u+v)K_2^+(v) = K_2^+(v)\mathcal{R}_{12}(u+v)K_1^+(u)R_{21}(v-u), \end{cases}$$

- Consider the following K**-matrices** ($\xi^{\pm}, \phi, \psi \in \mathbb{C}$):

$$K^-(u) = \begin{pmatrix} \xi^- + u - \eta/2 & 0 \\ 0 & \xi^- - u + \eta/2 \end{pmatrix},$$

$$K^+(u) = \begin{pmatrix} \xi^+ + u + \eta/2 & \psi(u+\eta/2) \\ \phi(u+\eta/2) & \xi^+ - u - \eta/2 \end{pmatrix}.$$

- The **transfer matrix** $\in \mathrm{End}(\mathcal{H})$

$$t(u) = \mathrm{tr}_a \left(K_a^+(u) L_{a\mathcal{L}}(u-\varepsilon_{\mathcal{L}}) \ldots L_{a1}(u-\varepsilon_1) K_a^-(u) L_{a1}(u+\varepsilon_1) \ldots L_{a\mathcal{L}}(u+\varepsilon_{\mathcal{L}}) \right)$$

satisfies $[t(u), t(v)] = 0 \;\; \forall u, v \in \mathbb{C} \;\; \Rightarrow$ can be used as a generating function for the conserved operators.

Constructing the conserved operators

Take the **quasi-classical limit** to construct the conserved operators:

$$\lim_{u\to\varepsilon_j}(u-\varepsilon_j)t(u)=\eta^2\tau_j+o(\eta^2).$$

Condition: for it to be well-defined the K-matrices have to satisfy

$$K^+(u)K^-(u)\to f(u)I \quad \text{as} \quad \eta\to 0. \qquad (\dagger)$$

Assume that parameters depend on η as follows:

$$\boxed{\xi^+=\xi+\eta\alpha,\quad \xi^-=-\xi+\eta\beta,\quad \psi=\eta\gamma,\quad \phi=\eta\lambda}$$

Then $(\dagger)$ is satisfied and the **conserved operators** are

$$\tau_j^*=\sum_{k\neq j}^{\mathcal{L}}\frac{4\varepsilon_j^2}{\varepsilon_j^2-\varepsilon_k^2}S_j^zS_k^z+\sum_{k\neq j}^{\mathcal{L}}\frac{2\varepsilon_j\varepsilon_k}{\varepsilon_j^2-\varepsilon_k^2}(S_j^+S_k^-+S_j^-S_k^+)+$$
$$+2(\alpha+\beta)S_j^z+\gamma\varepsilon_jS_j^+-\lambda\varepsilon_jS_j^-.$$

Constructing the Hamiltonian

$$\sum_{j=1}^{\mathcal{L}} \varepsilon_j^{-2}\tau_j^* = -2\sum_{j,k:j<k} \varepsilon_j^{-1}\varepsilon_k^{-1}(S_j^+S_k^- + S_j^-S_k^+) + 2(\alpha+\beta)\sum_{j=1}^{\mathcal{L}} \varepsilon_j^{-2}S_j^z + \\ + \gamma\sum_{j=1}^{\mathcal{L}} \varepsilon_j^{-1}S_j^+ - \lambda\sum_{j=1}^{\mathcal{L}} \varepsilon_j^{-1}S_j^- \equiv H'.$$

Making the change of variable $z_j = \varepsilon_j^{-1}$ we obtain

$$H' = 2(\alpha+\beta)\sum_{j=1}^{\mathcal{L}} z_j^2 S_j^z - 2\sum_{j,k:j<k} z_j z_k (S_j^+S_k^- + S_j^-S_k^+) + \gamma\sum_{j=1}^{\mathcal{L}} z_j S_j^+ - \lambda\sum_{j=1}^{\mathcal{L}} z_j S_j^-.$$

Set $\gamma = -\lambda$. Then $H = \frac{1}{2}GH'$ with $\alpha + \beta = G^{-1}$ and $\gamma = 2\Gamma G^{-1}$:

$$\boxed{H = \sum_{k=1}^{\mathcal{L}} z_k^2 S_k^z - G\sum_{k=1}^{\mathcal{L}}\sum_{j\neq k} z_k z_j S_k^+ S_j^- + \Gamma\sum_{k=1}^{\mathcal{L}} z_k\left(S_k^+ + S_k^-\right)}$$

The energy spectrum

Utilising the ODBA applied to the XXX Gaudin model[6] we obtain

- the **eigenvalues** of the conserved operators τ_j^*

$$\lambda_j^* = \sum_{k\neq j}^{\mathcal{L}} \frac{\varepsilon_j^2}{\varepsilon_j^2 - \varepsilon_k^2} - \sum_{i=1}^{\mathcal{L}} \frac{2\varepsilon_j^2}{\varepsilon_j^2 - v_i^2} - \alpha$$

- the **eigenvalues** of the Hamiltonian H (the **energy spectrum**)

$$E = (1+G)\sum_{i=1}^{\mathcal{L}} y_i - \frac{1}{2}\sum_{j=1}^{\mathcal{L}} z_j^2 + \Gamma^2 G^{-1} \sum_{i=1}^{\mathcal{L}} \frac{\prod_{j=1}^{\mathcal{L}}(1 - y_i z_j^{-2})}{\prod_{k\neq i}^{\mathcal{L}}(1 - y_i y_k^{-1})}$$

- subject to the **Bethe Ansatz Equations** $(k = 1, \ldots, \mathcal{L})$

$$1 + G^{-1} + \sum_{i\neq k}^{\mathcal{L}} \frac{2y_i}{y_i - y_k} - \sum_{l=1}^{\mathcal{L}} \frac{z_l^2}{y_k - z_l^2} = -\Gamma^2 G^{-2} \frac{1}{y_k} \frac{\prod_{l=1}^{\mathcal{L}}(1 - y_k z_l^{-2})}{\prod_{i\neq k}^{\mathcal{L}}(1 - y_k y_i^{-1})}$$

[6] Hao, Cao, Yang, Yang 2015

Thank you for your attention!

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(arXiv:1507.04068)