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RICHARDSON–GAUDIN MODELS FROM THE BOUNDARY QUANTUM INVERSE SCATTERING METHOD

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Abstract

Richardson–Gaudin models are a class of quantum integrable models connected to many physical systems, including pairing Hamiltonians from the theory of superconductivity. They can be obtained in the quasi-classical limit of the Quantum Inverse Scattering Method, which is based on an R-matrix and a Lax operator satisfying the Yang-Baxter equation. They can also be obtained from the Boundary Quantum Inverse Scattering Method, which relies on solutions of the reflection equations known as K-matrices. In this thesis we study these latter models systematically, explore the connections between them and investigate the interpretation of the "boundary".

First of all, we consider Richardson–Gaudin models obtained from the spin-1/2 $\mathfrak{su}(2)$ Boundary Quantum Inverse Scattering Method with *diagonal* K-matrices. We prove that the trigonometric boundary construction is equivalent to its rational limit, through a change of variables, rescaling, and a basis transformation. Moreover, we prove that the twisted-periodic and boundary constructions are equivalent in the trigonometric case, but not in the rational limit. Thus, including the "boundary" does not lead to a new model in this case.

Next, we investigate Richardson–Gaudin models obtained from the spin-1/2 $\mathfrak{su}(2)$ Boundary Quantum Inverse Scattering Method with *non-diagonal* K-matrices. Here the situation is different. The conserved operators in the boundary construction are no longer equivalent to the ones in the twisted-periodic construction. Also, the rational and the trigonometric boundary constructions are not equivalent. In the *rational* case this allows us to construct a generalisation of the $p+ip$ pairing Hamiltonian with external interaction terms. In the *trigonometric* case the expressions for the conserved operators involve several free parameters, which can be adjusted to construct a variety of Hamiltonians. This result offers opportunities for future investigations.

Finally, we study the case of the q-deformed bosonic Lax operator. This case is much more challenging than the case of the spin Lax operator. It is not straightforward to define the rational and quasi-classical limits of the bosonic Lax operator. Even after making modifications to the Lax operator for these limits to be well defined, it turns out that the limits do not commute. We state some open questions for future work.

Declaration by author

This thesis is composed of my original work, and contains no material previously published or written by another person except where due reference has been made in the text. I have clearly stated the contribution by others to jointly-authored works that I have included in my thesis.

I have clearly stated the contribution of others to my thesis as a whole, including statistical assistance, survey design, data analysis, significant technical procedures, professional editorial advice, and any other original research work used or reported in my thesis. The content of my thesis is the result of work I have carried out since the commencement of my research higher degree candidature and does not include a substantial part of work that has been submitted to qualify for the award of any other degree or diploma in any university or other tertiary institution. I have clearly stated which parts of my thesis, if any, have been submitted to qualify for another award.

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Publications during candidature

1. I. Lukyanenko, P. S. Isaac, and J. Links. On the boundaries of quantum integrability for the spin-1/2 Richardson–Gaudin system. *Nucl. Phys. B*, 866:364–398, 2014.

2. A. I. Bobenko, U. Hertrich-Jeromin, and I. Lukyanenko. Discrete constant mean curvature nets in space forms: Steiner's formula and Christoffel duality. *Discrete Comput. Geom.*, 52:612–629, 2014.

3. I. Lukyanenko, P. S. Isaac, and J. Links. An integrable case of the $p + ip$ pairing Hamiltonian interacting with its environment. *J. Phys. A: Math. Theor.*, 49:084001, 2016.

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The Technical Calculations	70%	15%	15%
The Drafting and Writing	50%	20%	30%

Contributions by others to the thesis

My supervisors, P. S. Isaac and J. Links, significantly contributed to the design of the project plan, directed me towards the relevant literature and helped to develop the techniques used in the thesis. They have also undertaken proofreading of the thesis.

Statement of parts of the thesis submitted to qualify for the award of another degree

None.

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Keywords

quantum integrable models, Yang–Baxter equation, algebraic Bethe Ansatz, Bethe Ansatz equations, boundary quantum inverse scattering method, Richardson–Gaudin models, $p + ip$ pairing Hamiltonian, off-diagonal Bethe Ansatz.

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Chapter 1

Introduction

Quantum integrability continues to be a fruitful area of research on the border between mathematics and physics. In mathematics it has led to the development of many new algebraic structures, such as quantum groups [Jim85, Dri87]. In physics, it has found multiple applications, e.g., in the study of ultrasmall superconducting grains [DS00, SDD$^+$00, vDR01], nuclear physics [DPS04], cold atoms [GBLZ08], quantum optics [Gar11] and quantum information [YN05, CW08, Tsa10].

In classical Hamiltonian mechanics, a system is said to be *integrable* if the number of independent conserved quantities is equal to the number of degrees of freedom of the system. If the number of independent conserved quantities is greater than the number of degrees of freedom of the system, a system is said to be *superintegrable*. In the case of a quantum system the definition of integrability is not as straightforward. For instance, the number of degrees of freedom is not always clearly determined. There are several approaches to quantum integrability (discussed in [CM11] and [Lar13]), but as yet there is no universally accepted definition.

We work in the context of the *Yang–Baxter integrability*, i.e., we refer to a model as *quantum integrable* if it can be constructed via the *Quantum Inverse Scattering Method* (QISM). The QISM is a powerful mathematical technique for the construction and solution of quantum integrable models, which was developed in the late 1970s in Leningrad by Faddeev, Kulish, Sklyanin, Takhtadzhan and others ([TF79, KS79, KS82, Fad95, Fad96]). The key ingredients of the QISM are an *R-matrix*, which is a solution of the *Yang–Baxter equation* [Bax72, Yan67], and a *Lax operator* satisfying the *RLL relation* (a version of the Yang–Baxter equation) together with the R-matrix. These ingredients are used to construct a one-parameter family of commuting transfer matrices, which in turn generate a set of mutually commuting conserved operators, including the Hamiltonian of the model.

An *exactly solvable model* is a model for which the eigenvalues and the eigenstates of the Hamiltonian can be exactly determined. In 1931 Bethe developed a method (that now goes under the name of the *co-ordinate Bethe Ansatz*) for deriving the exact solution of the Heisenberg XXX spin chain [Bet31]. The method turns the problem of finding the spectrum of the Hamiltonian into solving a system of coupled equations, referred to as the *Bethe Ansatz Equations* (BAE). Subsequently this method was developed further by others and various forms of it appeared. We will predominantly use the algebraic form, called the *algebraic Bethe Ansatz*, which was developed in parallel with and incorporated into the QISM. In Chapter 4 we will also use the recently developed *off-diagonal Bethe Ansatz* method [WYCS15].

In this thesis we study Richardson–Gaudin models, which came to prominence in some part due to connections with pairing Hamiltonians from the BCS theory of superconductivity (Bardeen, Cooper and Schrieffer in 1957 [BCS57]). In 1963 Richardson announced an exact solution of the reduced BCS Hamiltonian, also known as the s-wave pairing Hamiltonian [Ric63]. This result was further developed in a series of papers by Richardson and Sherman [RS64, Ric65, Ric66, Ric67, Ric68]. Richardson's approach is akin to the co-ordinate Bethe Ansatz [Bet31], which does not rely on a solution of the Yang–Baxter equation. In 1976 Gaudin provided, also without utilising the Yang–Baxter equation, a general algebraic formulation for constructing integrable systems related to the $\mathfrak{su}(2)$ Lie algebra [Gau76]. In doing so he obtained the exact solution for a class of interacting spin models, referred to as Gaudin models. Gaudin pointed out that these have a similar form of the BAE as those of Richardson's solution, but no further connection was established at that point.

Independent of the works by Richardson and Gaudin, in 1997 Cambiaggio, Rivas and Saraceno determined a set of conserved operators for the s-wave pairing Hamiltonian [CRS97], in the context of nuclear physics. Shortly after, experiments conducted on metallic nanograins (reviewed in [vDR01]) led to the rediscovery [DS00] of Richardson's hitherto little-known exact solution. In 2001 Amico et al [ALO01] and Dukelsky at al [DES01] independently presented a trigonometric generalisation of Richardson's model using Gaudin's method, which connects back to Richardson's model in the rational limit. Following a review [DPS04] it has become commonplace to refer to these models as *Richardson–Gaudin models*. The elliptic case (see, e.g., [ST96, ED15]) is more challenging, since it breaks $\mathfrak{u}(1)$ symmetry leading to non-conservation of particle number. In this thesis we will focus on the rational and trigonometric constructions, leaving the elliptic case for future work.

Shortly after the development of the QISM in the late 1970s and early 1980s it was realised that Gaudin models can be viewed as the *quasi-classical limit* of inhomogeneous spin chains (see Chapter 13.2 of [Gau83] and [Skl89, HKW92, Bab93, BF94]). However these works did not make connection with pairing Hamiltonians, and it was only after the rediscovery of Richardson's solution that the correspondence was realised in full. In particular, it was clarified that Richardson's solution for the s-wave model, and the conserved operators, may be obtained as the quasi-classical limit of the twisted-periodic rational $\mathfrak{su}(2)$ transfer matrix of the QISM with generic inhomogeneities [AFF01, vDP02, ZLMG02, Ovc03], and that the trigonometric analogue is related to the $p + ip$ pairing Hamiltonian [Skr09, ILSZ09, DIL+10, RDO10].

In 1988 Sklyanin proposed the *Boundary Quantum Inverse Scattering Method* (BQISM) [Skl88]. Based on the Yang–Baxter equation and the *reflection equations* [Che84], this formalism permits the construction of one-dimensional quantum systems with integrable boundary conditions, and the derivation of associated exact Bethe Ansatz solutions. The boundary conditions are encoded in the left and right *reflection matrices*, or K*-matrices*, satisfying the reflection equations. The examples of the XXZ and XYZ spin chains, the non-linear Schrödinger equation, and the Toda chain are discussed in [Skl88]. The method has been widely applied to the construction and analyses of one-dimensional quantum models with integrable boundaries, and related mathematical structures, for more than two decades, e.g., [KS92, AAC+03, Gal08, FSW08, FGSW11, Nic12, BCR13, dGLR13, PLS13, FKN14]. The K-matrices for the XXX, XXZ and XYZ spin chains were classified in [dVGR94].

The quasi-classical limit of the BQISM was studied by Sklyanin in [Skl87], prior to his more well-known publication [Skl88]. Adopting this approach, several authors have implemented constructions to produce generalised versions of Richardson–Gaudin systems [Hik95, LAH+02, YZG04, Skr07, Skr10, AMN13]. In-depth analyses however, including implications for formulating new pairing Hamiltonians, appear to have not been widely undertaken. This thesis aims to fill this gap, motivated by a wish to understand the interpretation of the "boundaries" in the Richardson–Gaudin context.

In the first part of the thesis [LIL14] (incorporated as a part of Chapter 2 and Chapter 3) we study Richardson–Gaudin models obtained from the spin-1/2 $\mathfrak{su}(2)$ BQISM with diagonal K-matrices. We introduce a generalised version of Sklyanin's construction using the trigonometric six-vertex solution of the Yang–Baxter equation which extends the approach of Karowski and Zapletal [KZ94] to include inhomogeneities in the transfer matrix. The algebraic Bethe Ansatz is applied to determine the transfer matrix eigenvalues and associated BAE. This formulation is dependent on a parameter ρ such that Sklyanin's construction is obtained by setting $\rho = 0$. In the limit $\rho \to \infty$ the twisted-periodic transfer matrix is recovered. We refer to this as the *attenuated limit*, since it has the effect of collapsing the double-row transfer matrix to the single-row transfer matrix. We also discuss the rational limit, and illustrate the general framework for the well-known case of the Heisenberg XXZ and XXX models.

Next, we turn our attention to a detailed analysis of the quasi-classical limit of this construction. We initially study the BAE in this limit, and establish that several equivalences emerge through appropriately chosen changes of variables. We then show that the same equivalences extend to the conserved operators of the system by identifying appropriate rescalings and basis transformations. For completeness, we confirm that the equivalences hold at the level of eigenvalue expressions for the conserved operators.

The conclusion from our calculations is that the boundary construction for the spin-1/2 case, with the use of diagonal solutions of the reflection equations, does not extend the class of conserved operators beyond results obtained from the twisted-periodic construction. All results for the BAE, the conserved operators, and their eigenvalues can be mapped back, through appropriate changes of variables (and also rescalings and basis transformations in the case of the conserved operators) to analogous quantities obtained from the twisted-periodic formulation. Nonetheless, some surprising features are uncovered. We prove that the trigonometric BQISM construction in the quasi-classical limit is equivalent to its rational limit. Moreover, we prove that the twisted-periodic and boundary constructions are equivalent in the trigonometric case, but not in the rational limit.

In the second part of the thesis [LIL16] (incorporated as Chapter 4) we consider the situation when the K-matrices are non-diagonal. We start with the rational BQISM with generic non-diagonal K-matrices and derive the formulae for the conserved operators in the quasi-classical limit. A similar construction has been already studied in [AMS14] and [AMRS15]. In contrast to these papers we consider a more general quasi-classical expansion and also prove that the two families of conserved operators derived in [AMS14, AMRS15] are, in fact, equivalent. Next, assuming that one of the K-matrices is diagonal (this can almost always be achieved by a basis transformation) we simplify the expressions for the conserved operators. A linear combination of these operators gives an integrable extension of the $p + ip$ Hamiltonian with external interaction terms of a patricular form. (The integrability and exact solvability of the isolated $p + ip$ pairing model was established previously in [ILSZ09].) These interaction terms allow for the exchange of particles between the system and its environment and, thus, break the $\mathfrak{u}(1)$ invariance associated with conservation of particle number.

It is well known that broken $\mathfrak{u}(1)$ symmetry causes some technical difficulties in applying the algebraic Bethe Ansatz, in particular, due to the absence of an obvious reference state. Recently, a systematic method, referred to as the *off-diagonal Bethe Ansatz*, has been proposed for solving these models [CYSW13a, CYSW13b, CYSW13c, CCY$^+$14]. It has since been applied to several long-standing problems [LCY$^+$14, ZCY$^+$14, HCL$^+$14] and the results has been summarised in the book by Wang et al. [WYCS15]. In [HCYY15] this method has been applied to the XXX Gaudin model with generic open boundaries. Based on this result we derive the formulae for the eigenvalues of the conserved operators, the corresponding BAE and the energy spectrum (the eigenvalues of the Hamiltonian).

To further develop the project on Richardson–Gaudin models from the BQISM, in Chapter 5 we consider the most challenging case, based on the trigonometric solution of the Yang–Baxter equation and generic non-diagonal trigonometric K-matrices. First of all, we calculate the conserved operators in the quasi-classical limit and show how the diagonal and rational limits indeed connect this case to the cases considered previously. The expressions for the conserved operators in this case are quite cumbersome and involve several free parameters. By adjusting these parameters we can construct various Hamiltonians. For a particular choice of parameters the conserved operators look very similar to *elliptic* Gaudin Hamiltonians, obtained in the quasi-classical limit of the XYZ spin-1/2 spin chain [ST96, ED15]. This suggests an equivalence between the trigonometric boundary construction and elliptic periodic construction, similar to the connection between the rational boundary construction and the trigonometric twisted-periodic construction, which we established previously. Further investigation is required to explore this connection.

Finally, in Chapter 6 we investigate the models based on the *bosonic* Lax operator, which is used in applying the QISM to various physical models, including Tavis–Cummings models in quantum optics [BBT96] and the two-site Bose–Hubbard model [ESKS91, ESSE92, EKS93, ZLMG03, LH06, LFTS06, TY13, SFR13]. First of all, we revise the QISM procedure in the periodic case and, after that, we include the boundary by applying the BQISM construction. Surprisingly, this does not lead to increasing of the number of independent conserved operators. Thus, we proceed to consider the case of the q-deformed Lax operator, introduced in [Kun07a], to see if the situation is different there.

This case is much more challenging than the case of the trigonometric spin-1/2 Lax operator, which had both rational and quasi-classical limits well-defined and mutually commuting. In the case of the q-deformed bosonic Lax operator it is not straightforward to define the rational and quasi-classical limits. We show how to modify the Lax operator in order for these limits to be well-defined. Then, we investigate whether the limits commute. While the quasi-classical limit of the rational limit is well-defined, there appears to be no obvious way to define the rational limit of the quasi-classical limit. We show that this difficulty is also present at the level of the BAE. Further investigation is needed to reveal the full implications of this. In an attempt to overcome this problem we consider an alternative Lax operator given by a special form of the monodromy matrix, for which both rational and quasi-classical limits are well-defined. Unfortunately this leads to other technical difficulties. We plan to continue this direction of research in future. The main motivation is to ultimately investigate the models obtained by *combining* the spin Lax operator and the bosonic Lax operator in the (B)QISM construction, e.g., see [AFOR07, AFOW10]. From the physical point of view, this has a potential application in the study of matter-radiation models [Kun04, Kun05, Kun06, Kun07b] and problems related to integrability of the Rabi model [Bra11, BZ15].

Chapter 2

Preliminaries

The purpose of this chapter is to provide some background knowledge for this thesis. First of all, we briefly review the QISM and the algebraic Bethe Ansatz (for more detailed information see [Fad95]). We also introduce the spin-1/2 Richardson–Gaudin models obtained in the quasi-classical limit from the twisted-periodic QISM construction. Next, we review Sklyanin's BQISM construction [Skl88] (for the trigonometric, spin-1/2 $\mathfrak{su}(2)$ case) and introduce a generalisation of the BQISM depending on an additional complex parameter ρ, so that setting $\rho = 0$ it gives back Sklyanin's formulation and in the limit as $\rho \to \infty$, which we call the *attenuated limit*, we obtain the twisted-periodic QISM. We explain in detail how the attenuated limit works for the original trigonometric construction and for its rational limit. The connections we will obtain are summarised in Figure 2.1 below. Finally, we discuss how the Heisenberg model can be constructed as a special case of this generalised approach and look at the analogous connections there (summarised in Figure 2.2 below).

2.1 Quantum Inverse Scattering Method

Throughout this thesis we fix a vector space $V = \mathbb{C}^2$. The key ingredient of the QISM is the *R-matrix*, which is an invertible operator $R(u) \in \mathrm{End}(V \otimes V)$ depending on the spectral parameter $u \in \mathbb{C}$ and satisfying the *Yang–Baxter equation* [Yan67, Bax72]

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v). \tag{2.1}$$

It is an operator equation in $\mathrm{End}(V \otimes V \otimes V)$, with the subscripts indicating the spaces in which the corresponding R-matrix acts non-trivially.

There are three standard classes of R-matrices: elliptic, trigonometric and rational. In this thesis we will only consider trigonometric and rational cases. Let us introduce the trigonometric[1] R-matrix associated with the XXZ model [Bax72]:

$$R(u) = \frac{1}{\sinh(u+\eta)} \begin{pmatrix} \sinh(u+\eta) & 0 & 0 & 0 \\ 0 & \sinh u & \sinh\eta & 0 \\ 0 & \sinh\eta & \sinh u & 0 \\ 0 & 0 & 0 & \sinh(u+\eta) \end{pmatrix}, \tag{2.2}$$

where $\eta \in \mathbb{C}$ is the quasi-classical parameter. Note that (2.2) is symmetric, i.e., $R_{12}(u) = R_{21}(u)$ and satisfies the unitarity property: $R_{12}(u)R_{12}(-u) = I \otimes I$.

The rational R-matrix

$$R(u) = \frac{1}{u+\eta}(uI \otimes I + \eta P) = \frac{1}{u+\eta} \begin{pmatrix} u+\eta & 0 & 0 & 0 \\ 0 & u & \eta & 0 \\ 0 & \eta & u & 0 \\ 0 & 0 & 0 & u+\eta \end{pmatrix}, \tag{2.3}$$

where P is the permutation operator, can be obtained from (2.2) by introducing a parameter $\nu \in \mathbb{C}$ as a scaling factor in the argument of the hyperbolic functions, and using

$$\lim_{\nu \to 0} \frac{\sinh(\nu x)}{\nu} = x. \tag{2.4}$$

We refer to this procedure as taking the *rational limit*.

The construction goes as follows. The *quantum space* of the system (i.e., the Hilbert space of states) is constructed as a tensor product of local vector spaces V_l (at the moment we do not specify these):

$$\mathcal{H} = \bigotimes_{l=1}^{\mathcal{L}} V_l. \tag{2.5}$$

For each label l in the above tensor product (2.5), we introduce an object called the *Lax operator* $L_{al}(u) \in \mathrm{End}(V_a \otimes V_l)$ satisfying the *RLL relation*

$$R_{ab}(u-v)L_{al}(u)L_{bl}(v) = L_{bl}(v)L_{al}(u)R_{ab}(u-v), \tag{2.6}$$

where the auxiliary spaces V_a and V_b are copies of V.

[1]While it is conventional to refer to the R-matrix as trigonometric, for convenience we adopt the hyperbolic parametrisation.

We need to introduce another auxiliary object, called the *monodromy matrix*. It is an invertible operator acting on $V_a \otimes \mathcal{H}$ constructed as follows:

$$T_a(u) = M_a L_{a\mathcal{L}}(u - \varepsilon_{\mathcal{L}}) \cdots L_{a2}(u - \varepsilon_2) L_{a1}(u - \varepsilon_1), \text{ where } M = \begin{pmatrix} e^{-\eta\gamma} & 0 \\ 0 & e^{\eta\gamma} \end{pmatrix}. \qquad (2.7)$$

The parameters $\varepsilon_j \in \mathbb{C}$ are known as *inhomogeneities*. These are typically set to be zero in the construction of one-dimensional quantum lattice models, but are retained as generic parameters in Richardson–Gaudin models. The matrix M encodes the *twisted-periodic boundary conditions*.

One can write the monodromy matrix (2.7) as an operator valued 2×2-matrix in the auxiliary space

$$T_a(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix},$$

where the entries are operators acting in the quantum space $\mathcal{H}$. Using an induction argument and the fact that

$$R_{ab}(u - v) M_a M_b = M_b M_a R_{ab}(u - v)$$

one can prove that (2.7) satisfies the following equation in $\mathrm{End}(V_a \otimes V_b \otimes \mathcal{H})$:

$$R_{ab}(u - v) T_a(u) T_b(v) = T_b(v) T_a(u) R_{ab}(u - v). \qquad (2.8)$$

The *transfer matrix* is an operator acting in the quantum space $\mathcal{H}$, given by

$$t(u) = \mathrm{tr}_a(T_a(u)) = A(u) + D(u). \qquad (2.9)$$

The equation (2.8) implies that the transfer matrices given by (2.9) satisfy

$$[t(u), t(v)] = 0 \text{ for all } u, v \in \mathbb{C}. \qquad (2.10)$$

This commutativity property is crucial, as it allows us to use the transfer matrix for construction of quantum integrable models. For example, consider the expansion of $t(u)$ in powers of u:

$$t(u) = \sum_{j=-\infty}^{\infty} C_j u^j.$$

From (2.10) it follows that $[C_j, C_k] = 0$ for all j, k. Thus, the transfer matrix (2.9) generates a family of mutually commuting operators (which generically will be simultaneously diagonalisable). Then any function $H = f(\{C_j\})$ of C_j can be taken as the Hamiltonian a quantum integrable model, for which $\{C_j\}$ will constitute a set of mutually commuting conserved operators.

2.2 Algebraic Bethe Ansatz

Here we outline the *algebraic Bethe Ansatz* method which allows to exactly solve the constructed model (i.e., find the eigenstates and the eigenvalues of the Hamiltonian). By writing (2.8) in terms of 4×4-matrices in $V_a \otimes V_b$, we find, among others, the following commutation relations for $A(u), C(u)$ and $D(u)$:

$$\begin{aligned} &[C(u), C(v)] = 0, \\ A(u)C(v) &= \frac{\sinh(u - v + \eta)}{\sinh(u - v)} C(v)A(u) - \frac{\sinh \eta}{\sinh(u - v)} C(u)A(v), \\ D(u)C(v) &= \frac{\sinh(u - v - \eta)}{\sinh(u - v)} C(v)D(u) + \frac{\sinh \eta}{\sinh(u - v)} C(u)D(v). \end{aligned} \tag{2.11}$$

In order to construct the eigenstates of $t(u)$ we start with a *reference state* $\Omega \in \mathcal{H}$ satisfying

$$B(u)\Omega = 0, \;\; A(u)\Omega = a(u)\Omega, \;\; D(u)\Omega = d(u)\Omega, \;\; C(u)\Omega \neq 0, \tag{2.12}$$

where $a(u)$ and $d(u)$ are scalar functions, so that Ω is an eigenstate for $A(u)$ and $D(u)$ simultaneously and, hence, also for $t(u) = A(u) + D(u)$.

Remark 2.1. *The reference state Ω is an analogue to the "lowest weight" state in the representation theory of $\mathfrak{sl}(2)$ and $\mathfrak{gl}(2)$.*

We look for other eigenstates in the form

$$\Phi = \Phi(v_1, \ldots, v_N) = C(v_1) \cdots C(v_N)\Omega. \tag{2.13}$$

Using the commutation relations (2.11) one can see that (2.13) is an eigenstate of $t(u)$ with the eigenvalue

$$\Lambda(u, v_1, \ldots, v_N) = a(u) \prod_{k=1}^{N} \frac{\sinh(u - v_k + \eta)}{\sinh(u - v_k)} + d(u) \prod_{k=1}^{N} \frac{\sinh(u - v_k - \eta)}{\sinh(u - v_k)} \tag{2.14}$$

if and only if $\Phi \neq 0$ and parameters $\{v_k\}$ satisfy the BAE

$$\frac{a(v_k)}{d(v_k)} = \prod_{i\neq k}^{N} \frac{\sinh(v_k - v_i - \eta)}{\sinh(v_k - v_i + \eta)}, \quad k = 1, \dots, N. \tag{2.15}$$

Remark 2.2. *In this thesis we will use a simplified notation for sums and products, i.e., we write* $\prod_{i\neq k}^{N}$ *instead of writing* $\prod_{\substack{i=1\\ i\neq k}}^{N}$ *and* $\sum_{j\neq k}^{\mathcal{L}}$ *instead of writing* $\sum_{\substack{j=1\\ j\neq k}}^{\mathcal{L}}$.

In the rational limit (2.4) from (2.14) and (2.15) we obtain

$$\Lambda(u, v_1, \dots, v_N) = a(u) \prod_{k=1}^{N} \frac{u - v_k + \eta}{u - v_k} + d(u) \prod_{k=1}^{N} \frac{u - v_k - \eta}{u - v_k},$$

$$\frac{a(v_k)}{d(v_k)} = \prod_{i\neq k}^{N} \frac{v_k - v_i - \eta}{v_k - v_i + \eta}, \quad k = 1, \dots, N,$$

where the functions $a(u)$ and $d(u)$ are in the rational limit.

2.3 Spin-1/2 Richardson–Gaudin models

Richardson–Gaudin models are obtained by taking the quasi-classical limit ($\eta \to 0$) from the QISM. First of all, let us specify the QISM construction for this case.

Here each local space V_l in the tensor product (2.5) is a spin-1/2 representation space $V = \mathbb{C}^2$ for the $\mathfrak{su}(2)$ Lie algebra spanned by the spin operators S_l^-, S_l^+, S_l^z (indices indicate in which space the corresponding operator acts non-trivially), which for the spin-1/2 case are given by

$$S^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ S^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ S^z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.16}$$

and satisfy the $\mathfrak{su}(2)$ commutation relations

$$[S^z, S^\pm] = \pm S^\pm, \ [S^+, S^-] = 2S^z.$$

For each label l in the tensor product (2.5), we introduce the trigonometric spin-1/2 Lax operator obtained as a scaling of the (shifted) R-matrix (2.2):

$$L_{al}(u) = \frac{\sinh(u+\eta/2)}{\sinh u} R_{al}(u-\eta/2) = \tag{2.17}$$
$$= \frac{1}{\sinh u}\begin{pmatrix} \sinh(u+\eta/2) & 0 & 0 & 0 \\ 0 & \sinh(u-\eta/2) & \sinh\eta & 0 \\ 0 & \sinh\eta & \sinh(u-\eta/2) & 0 \\ 0 & 0 & 0 & \sinh(u+\eta/2) \end{pmatrix}.$$

Remark 2.3. *The RLL relation (2.6) follows automatically from the Yang–Baxter equation (2.1).*

Remark 2.4. *It is easy to check that the operator (2.17) can be written as follows in terms of the spin operators (2.16):*

$$L_{al}(u) = \frac{1}{\sinh u}\begin{pmatrix} \sinh u \cosh\frac{\eta}{2} + 2\cosh u \sinh\frac{\eta}{2} S_l^z & \sinh\eta S_l^- \\ \sinh\eta S_l^+ & \sinh u\cosh\frac{\eta}{2} - 2\cosh u \sinh\frac{\eta}{2} S_l^z \end{pmatrix}.$$

Remark 2.5. *In the rational limit the Lax operator (2.17) reduces to*[2]

$$L_{al}(u) = \frac{1}{u}\begin{pmatrix} u+\eta S_l^z & \eta S_l^- \\ \eta S_l^+ & u - \eta S_l^z \end{pmatrix}. \tag{2.18}$$

Note that the state $\Omega = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{\otimes \mathcal{L}}$ satisfies the properties of a reference state (2.12) and

$$L_{al}(u-\varepsilon_l)\begin{pmatrix} 0 \\ 1 \end{pmatrix}_l = \frac{1}{\sinh(u-\varepsilon_l)}\begin{pmatrix} \sinh\left(u-\varepsilon_l-\frac{\eta}{2}\right) & 0 \\ * & \sinh\left(u-\varepsilon_l+\frac{\eta}{2}\right) \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}_l,$$

where we follow the tradition that $*$ denotes an operator which does not need to be known to continue calculations. Thus, we can calculate the expressions for $a(u)$ and $d(u)$:

$$a(u) = e^{-\eta\gamma}\prod_{l=1}^{\mathcal{L}} \frac{\sinh(u-\varepsilon_l-\eta/2)}{\sinh(u-\varepsilon_l)}, \quad d(u) = e^{\eta\gamma}\prod_{l=1}^{\mathcal{L}} \frac{\sinh(u-\varepsilon_l+\eta/2)}{\sinh(u-\varepsilon_l)}. \tag{2.19}$$

[2] We will not choose different notation for each Lax operator, but the one to be used will be specified at the beginning of each chapter.

Substituting (2.19) into (2.14) and (2.15) we obtain an explicit formula for the eigenvalues

$$\begin{aligned}\Lambda(u, v_1, \dots, v_N) = e^{-\eta\gamma} \prod_{l=1}^{\mathcal{L}} \frac{\sinh(u - \varepsilon_l - \eta/2)}{\sinh(u - \varepsilon_l)} \prod_{k=1}^{N} \frac{\sinh(u - v_k + \eta)}{\sinh(u - v_k)} + \\ + e^{\eta\gamma} \prod_{l=1}^{\mathcal{L}} \frac{\sinh(u - \varepsilon_l + \eta/2)}{\sinh(u - \varepsilon_l)} \prod_{k=1}^{N} \frac{\sinh(u - v_k - \eta)}{\sinh(u - v_k)}\end{aligned} \tag{2.20}$$

and for the BAE

$$e^{-2\eta\gamma} \prod_{l=1}^{\mathcal{L}} \frac{\sinh(v_k - \varepsilon_l - \eta/2)}{\sinh(v_k - \varepsilon_l + \eta/2)} = \prod_{i \neq k}^{N} \frac{\sinh(v_k - v_i - \eta)}{\sinh(v_k - v_i + \eta)}, \quad k = 1, \dots, N. \tag{2.21}$$

Now we take the quasi-classical limit in order to obtain the Richardson–Gaudin models, which involves expanding in powers of η as $\eta \to 0$ and taking the first non-trivial term in the expansion.

2.3.1 Bethe Ansatz Equations

We start by expanding the BAE (2.21) in powers of η. The expansion up to the first order of the left hand side gives

$$\begin{aligned}&e^{-2\eta\gamma} \prod_{l=1}^{\mathcal{L}} \frac{\sinh(v_k - \varepsilon_l - \eta/2)}{\sinh(v_k - \varepsilon_l + \eta/2)} = \\ &= e^{-2\eta\gamma} \prod_{l=1}^{\mathcal{L}} \frac{\sinh(v_k - \varepsilon_l)\cosh\frac{\eta}{2} - \cosh(v_k - \varepsilon_l)\sinh\frac{\eta}{2}}{\sinh(v_k - \varepsilon_l)\cosh\frac{\eta}{2} + \cosh(v_k - \varepsilon_l)\sinh\frac{\eta}{2}} = \\ &= (1 - 2\eta\gamma) \prod_{l=1}^{\mathcal{L}} \frac{\sinh(v_k - \varepsilon_l) - \frac{\eta}{2}\cosh(v_k - \varepsilon_l)}{\sinh(v_k - \varepsilon_l) + \frac{\eta}{2}\cosh(v_k - \varepsilon_l)} + \mathcal{O}(\eta^2) = \\ &= (1 - 2\eta\gamma) \prod_{l=1}^{\mathcal{L}} \frac{1 - \frac{\eta}{2}\coth(v_k - \varepsilon_l)}{1 + \frac{\eta}{2}\coth(v_k - \varepsilon_l)} + \mathcal{O}(\eta^2) = \\ &= (1 - 2\eta\gamma) \prod_{l=1}^{\mathcal{L}} \big(1 - \eta\coth(v_k - \varepsilon_l)\big) + \mathcal{O}(\eta^2) = \\ &= (1 - 2\eta\gamma) \left(1 - \eta \sum_{l=1}^{\mathcal{L}} \coth(v_k - \varepsilon_l)\right) + \mathcal{O}(\eta^2) = \\ &= 1 - 2\eta\gamma - \eta \sum_{l=1}^{\mathcal{L}} \coth(v_k - \varepsilon_l) + \mathcal{O}(\eta^2).\end{aligned}$$

The expansion up to the first order of the right hand side gives

$$\prod_{i\neq k}^{N}\frac{\sinh(v_k-v_i-\eta)}{\sinh(v_k-v_i+\eta)}=1-2\eta\sum_{i\neq k}^{N}\coth(v_k-v_i)+\mathcal{O}(\eta^2).$$

Putting these together we obtain that the BAE in the quasi-classical limit are given by

$$2\gamma+\sum_{l=1}^{\mathcal{L}}\coth(v_k-\varepsilon_l)=2\sum_{i\neq k}^{N}\coth(v_k-v_i),\quad k=1,\dots,N. \tag{2.22}$$

2.3.2 Conserved operators

In the quasi-classical limit, conserved operators τ_j are constructed as follows from the transfer matrix (2.9):

$$\lim_{u\to\varepsilon_j}(u-\varepsilon_j)t(u)=\eta^2\tau_j+\mathcal{O}(\eta^3). \tag{2.23}$$

It is easily verified (using the representation from Remark 2.4) that $L_{al}(u)$ given by (2.17) can be written as follows:

$$L_{al}(u)=I+\frac{\eta}{\sinh u}\ell_{al}(u)+\mathcal{O}(\eta^2), \tag{2.24}$$

where

$$\ell_{al}(u)=\begin{pmatrix}S_l^z\cosh u & S_l^-\\ S_l^+ & -S_l^z\cosh u\end{pmatrix}.$$

Remark 2.6. *Up to a scalar multiple, the trigonometric R-matrix* (2.2) *has the following quasi-classical expansion:*

$$R(u)=I+\eta r(u)+\mathcal{O}(\eta^2),$$

where

$$r(u)=\frac{1}{\sinh u}\begin{pmatrix}\cosh u & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & \cosh u\end{pmatrix}.$$

From the quasi-classical limit of the Yang–Baxter equation (2.1) *we obtain the classical Yang–Baxter equation for $r(u)$:*

$$[r_{12}(u-v),r_{13}(u)]+[r_{12}(u-v),r_{23}(v)]+[r_{13}(u),r_{23}(v)]=0.$$

Let us denote

$$\tilde{\ell}_{al}(u) = \frac{1}{\sinh u}\ell_{al}(u).$$

Then, from the quasi-classical limit of the RLL relation (2.6) *we obtain that* $\tilde{\ell}_{al}(u)$ *satisfies*

$$[r_{ab}(u-v), \tilde{\ell}_{al}(u)] + [r_{ab}(u-v), \tilde{\ell}_{bl}(v)] + [\tilde{\ell}_{al}(u), \tilde{\ell}_{bl}(v)] = 0.$$

The expansion of the twist matrix is given by

$$M = \begin{pmatrix} e^{-\eta\gamma} & 0 \\ 0 & e^{\eta\gamma} \end{pmatrix} = I - 2\eta\gamma S^z + \mathcal{O}(\eta^2). \tag{2.25}$$

Using the expressions (2.24) and (2.25) we can now calculate the expansion (2.23):

$$\begin{aligned}
&\lim_{u\to\varepsilon_j}(u-\varepsilon_j)t(u) = \\
&= \operatorname{tr}_a\Big[M_a L_{a\mathcal{L}}(\varepsilon_j-\varepsilon_{\mathcal{L}})\cdots L_{aj+1}(\varepsilon_j-\varepsilon_{j+1})\eta\ell_{aj}(0)L_{aj-1}(\varepsilon_j-\varepsilon_{j-1})\cdots L_{a1}(\varepsilon_j-\varepsilon_1)\Big] = \\
&= \eta\operatorname{tr}_a\Bigg[\big(I - 2\eta\gamma S^z_a + \mathcal{O}(\eta^2)\big)\Bigg(\ell_{aj}(0) + \eta\sum_{k=j+1}^{\mathcal{L}}\frac{\ell_{ak}(\varepsilon_j-\varepsilon_k)\ell_{aj}(0)}{\sinh(\varepsilon_j-\varepsilon_k)} + \\
&\qquad\qquad + \eta\sum_{k=1}^{j-1}\frac{\ell_{aj}(0)\ell_{ak}(\varepsilon_j-\varepsilon_k)}{\sinh(\varepsilon_j-\varepsilon_k)} + \mathcal{O}(\eta^2)\Bigg)\Bigg] = \\
&= \eta\operatorname{tr}_a\Bigg[\ell_{aj}(0) + \eta\Bigg(\sum_{k=j+1}^{\mathcal{L}}\frac{\ell_{ak}(\varepsilon_j-\varepsilon_k)\ell_{aj}(0)}{\sinh(\varepsilon_j-\varepsilon_k)} + \\
&\qquad\qquad + \sum_{k=1}^{j-1}\frac{\ell_{aj}(0)\ell_{ak}(\varepsilon_j-\varepsilon_k)}{\sinh(\varepsilon_j-\varepsilon_k)} - 2\gamma S^z_a\ell_{aj}(0)\Bigg) + \mathcal{O}(\eta^2)\Bigg].
\end{aligned}$$

Utilising

$$\begin{aligned}
&\operatorname{tr}_a(\ell_{aj}(0)) = 0, \quad \operatorname{tr}_a(S^z_a\ell_{aj}(0)) = S^z_j, \\
&\operatorname{tr}_a(\ell_{aj}(0)\ell_{ak}(\varepsilon_j-\varepsilon_k)) = \operatorname{tr}_a(\ell_{ak}(\varepsilon_j-\varepsilon_k)\ell_{aj}(0)) = \\
&\qquad\qquad = 2\cosh(\varepsilon_j-\varepsilon_k)S^z_kS^z_j + S^-_kS^+_j + S^+_kS^-_j,
\end{aligned}$$

we obtain from (2.23) a set of conserved operators for the trigonometric spin-1/2 Richardson–Gaudin model:

$$\tau_j = -2\gamma S^z_j + \sum_{k\neq j}^{\mathcal{L}}\frac{2\cosh(\varepsilon_j-\varepsilon_k)S^z_kS^z_j + S^-_kS^+_j + S^+_kS^-_j}{\sinh(\varepsilon_j-\varepsilon_k)}, \quad j = 1,\ldots,\mathcal{L}. \tag{2.26}$$

2.3.3 Eigenvalues

The eigenvalues λ_j of the conserved operators (2.26) are constructed as follows:

$$\lim_{u \to \varepsilon_j} (u - \varepsilon_j)\Lambda(u) = \eta^2 \lambda_j + \mathcal{O}(\eta^3). \tag{2.27}$$

To calculate this expansion let us first expand the expressions (2.19) for $a(u)$ and $d(u)$:

$$\begin{aligned}
a(u) &= e^{-\eta\gamma} \prod_{l=1}^{\mathcal{L}} \frac{\sinh(u - \varepsilon_l - \eta/2)}{\sinh(u - \varepsilon_l)} = \\
&= e^{-\eta\gamma} \prod_{l=1}^{\mathcal{L}} \frac{\sinh(u - \varepsilon_l)\cosh\frac{\eta}{2} - \cosh(u - \varepsilon_l)\sinh\frac{\eta}{2}}{\sinh(u - \varepsilon_j)} = \\
&= \left(1 - \eta\gamma + \frac{\eta^2\gamma^2}{2} + \mathcal{O}(\eta^3)\right) \prod_{l=1}^{\mathcal{L}} \left(1 + \frac{\eta^2}{8} - \frac{\eta}{2}\coth(u - \varepsilon_l) + \mathcal{O}(\eta^3)\right) = \\
&= \left(1 - \eta\gamma + \frac{\eta^2\gamma^2}{2} + \mathcal{O}(\eta^3)\right) \left(1 - \frac{\eta}{2}\sum_{l=1}^{\mathcal{L}} \coth(u - \varepsilon_l) + \frac{\eta^2}{8}\mathcal{L} + \right. \\
&\qquad \left. + \frac{\eta^2}{4}\sum_{l=1}^{\mathcal{L}}\sum_{k\neq l}^{\mathcal{L}} \coth(u - \varepsilon_l)\coth(u - \varepsilon_k) + \mathcal{O}(\eta^3)\right) = \\
&= 1 - \eta\gamma - \frac{\eta}{2}\sum_{l=1}^{\mathcal{L}} \coth(u - \varepsilon_l) + \\
&\quad + \eta^2\left[\frac{\mathcal{L}}{8} + \frac{\gamma^2}{2} + \frac{\gamma}{2}\sum_{l=1}^{\mathcal{L}} \coth(u - \varepsilon_l) + \frac{1}{4}\sum_{l=1}^{\mathcal{L}}\sum_{k\neq l}^{\mathcal{L}} \coth(u - \varepsilon_l)\coth(u - \varepsilon_k)\right] + \mathcal{O}(\eta^3).
\end{aligned}$$

Analogously,

$$\begin{aligned}
d(u) &= e^{\eta\gamma} \prod_{l=1}^{\mathcal{L}} \frac{u - \varepsilon_l + \eta/2}{u - \varepsilon_l} = 1 + \eta\gamma + \frac{\eta}{2}\sum_{l=1}^{\mathcal{L}} \coth(u - \varepsilon_l) + \\
&\quad + \eta^2\left[\frac{\mathcal{L}}{8} + \frac{\gamma^2}{2} + \frac{\gamma}{2}\sum_{l=1}^{\mathcal{L}} \coth(u - \varepsilon_l) + \frac{1}{4}\sum_{l=1}^{\mathcal{L}}\sum_{k\neq l}^{\mathcal{L}} \coth(u - \varepsilon_l)\coth(u - \varepsilon_k)\right] + \mathcal{O}(\eta^3).
\end{aligned}$$

Now we can compute the limits

$$\lim_{u \to \varepsilon_j} (u - \varepsilon_j)a(u) = -\frac{\eta}{2} + \frac{\eta^2}{2}\left(\gamma + \frac{1}{2}\sum_{k\neq j}^{\mathcal{L}} \coth(\varepsilon_j - \varepsilon_k)\right) + \mathcal{O}(\eta^3),$$

$$\lim_{u\to\varepsilon_j}(u-\varepsilon_j)d(u)=\frac{\eta}{2}+\frac{\eta^2}{2}\left(\gamma+\frac{1}{2}\sum_{k\neq j}^{\mathcal{L}}\coth(\varepsilon_j-\varepsilon_k)\right)+\mathcal{O}(\eta^3).$$

Then, from (2.14) we obtain

$$\begin{aligned}
&\lim_{u\to\varepsilon_j}(u-\varepsilon_j)\Lambda(u)=\\
&=\lim_{u\to\varepsilon_j}(u-\varepsilon_j)a(u)\prod_{i=1}^{N}\frac{\sinh(\varepsilon_j-v_i+\eta)}{\sinh(\varepsilon_j-v_i)}+\lim_{u\to\varepsilon_j}(u-\varepsilon_j)d(u)\prod_{i=1}^{N}\frac{\sinh(\varepsilon_j-v_i-\eta)}{\sinh(\varepsilon_j-v_i)}=\\
&=\left[-\frac{\eta}{2}+\frac{\eta^2}{2}\left(\gamma+\frac{1}{2}\sum_{k\neq j}^{\mathcal{L}}\coth(\varepsilon_j-\varepsilon_k)\right)+\mathcal{O}(\eta^3)\right]\left(1+\eta\sum_{i=1}^{N}\coth(\varepsilon_j-v_i)+\mathcal{O}(\eta^2)\right)+\\
&\quad+\left[\frac{\eta}{2}+\frac{\eta^2}{2}\left(\gamma+\frac{1}{2}\sum_{k\neq j}^{\mathcal{L}}\coth(\varepsilon_j-\varepsilon_k)\right)+\mathcal{O}(\eta^3)\right]\left(1-\eta\sum_{i=1}^{N}\coth(\varepsilon_j-v_i)+\mathcal{O}(\eta^2)\right)=\\
&=-\eta^2\sum_{i=1}^{N}\coth(\varepsilon_j-v_i)+\frac{\eta^2}{2}\sum_{k\neq j}^{\mathcal{L}}\coth(\varepsilon_j-\varepsilon_k)+\gamma\eta^2+\mathcal{O}(\eta^3).
\end{aligned}$$

Thus, we obtain the eigenvalues of the conserved operators (2.26) from (2.27):

$$\lambda_j=\gamma+\frac{1}{2}\sum_{k\neq j}^{\mathcal{L}}\coth(\varepsilon_j-\varepsilon_k)-\sum_{i=1}^{N}\coth(\varepsilon_j-v_i),\quad j=1,\ldots,\mathcal{L}. \tag{2.28}$$

2.3.4 Rational limit

Here we describe how to apply the rational limit (2.4) to the trigonometric expressions (2.22), (2.26) and (2.28).

- *BAE.*
  Introduce the parameter ν into (2.22) as follows:

$$\frac{2\gamma}{\nu}+\sum_{l=1}^{\mathcal{L}}\coth\big(\nu(v_k-\varepsilon_l)\big)=2\sum_{i\neq k}^{N}\coth\big(\nu(v_k-v_i)\big).$$

  Then multiply through by ν and consider $\nu\to 0$. Using $\lim_{\nu\to 0}\big(\nu\coth(\nu x)\big)=\frac{1}{x}$ we obtain the following BAE in the quasi-classical limit:

$$2\gamma+\sum_{l=1}^{\mathcal{L}}\frac{1}{v_k-\varepsilon_l}=\sum_{i\neq k}^{N}\frac{2}{v_k-v_i},\quad k=1,\ldots,N. \tag{2.29}$$

Similarly, we obtain conserved operators and their eigenvalues in the rational limit:

- *Conserved operators.*

$$\tau_j = -2\gamma S_j^z + \sum_{k\neq j}^{\mathcal{L}} \frac{2S_k^z S_j^z + S_k^- S_j^+ + S_k^+ S_j^-}{\varepsilon_j - \varepsilon_k}, \quad j = 1, \ldots, \mathcal{L}. \tag{2.30}$$

- *Eigenvalues.*

$$\lambda_j = \gamma + \frac{1}{2}\sum_{k\neq j}^{\mathcal{L}} \frac{1}{\varepsilon_j - \varepsilon_k} - \sum_{i=1}^{N} \frac{1}{\varepsilon_j - v_i}, \quad j = 1, \ldots, \mathcal{L}. \tag{2.31}$$

Remark 2.7. *In what follows we will denote the expressions (2.22), (2.26), (2.28) collectively as **Trig. QISM** and the rational expressions derived in Section 2.3.4 as **Rat. QISM**.*

2.4 Boundary Quantum Inverse Scattering Method

In the BQISM framework the boundary conditions are encoded in the left and right *reflection matrices*, or K*-matrices*, $\check{K}^-(u)$ and $\check{K}^+(u) \in \mathrm{End}(V)$, which satisfy the following *reflection equations* in $\mathrm{End}(V \otimes V)$ [Che84]:

$$R_{12}(u-v)\check{K}_1^-(u)R_{21}(u+v)\check{K}_2^-(v) = \check{K}_2^-(v)R_{12}(u+v)\check{K}_1^-(u)R_{21}(u-v), \tag{2.32a}$$

$$\begin{aligned} &R_{12}(v-u)\check{K}_1^+(u)R_{21}(-u-v-2\eta)\check{K}_2^+(v) = \\ &\qquad = \check{K}_2^+(v)R_{12}(-u-v-2\eta)\check{K}_1^+(u)R_{21}(v-u). \end{aligned} \tag{2.32b}$$

Let us introduce the monodromy matrix as (2.7) in the twisted-periodic case, but without the twist M (i.e., setting $\gamma = 0$),

$$T_a(u) = L_{a\mathcal{L}}(u-\varepsilon_{\mathcal{L}}) \cdots L_{a1}(u-\varepsilon_1), \tag{2.33}$$

where $L(u)$ is the trigonometric spin-1/2 Lax operator (2.17) and $\varepsilon_j \in \mathbb{C}$ are inhomogeneity parameters. Introduce the *dual monodromy matrix* as

$$\tilde{T}_a(u) = L_{a1}(u+\varepsilon_1+\eta) \cdots L_{a\mathcal{L}}(u+\varepsilon_{\mathcal{L}}+\eta). \tag{2.34}$$

Note that the Lax operator (2.17) satisfies $L_{aj}(u)L_{aj}(\eta - u) \propto I$. Thus,

$$\tilde{T}_a(u) \propto L_{a1}^{-1}(-u-\varepsilon_1)\cdots L_{a\mathcal{L}}^{-1}(-u-\varepsilon_{\mathcal{L}}) = T_a^{-1}(-u),$$

which implies, using (2.8), that (2.34) and (2.33) satisfy the following relations:

$$\tilde{T}_b(v)R_{ab}(u+v)T_a(u) = T_a(u)R_{ab}(u+v)\tilde{T}_b(v), \tag{2.35}$$

$$\tilde{T}_a(u)\tilde{T}_b(v)R_{ab}(v-u) = R_{ab}(v-u)\tilde{T}_b(v)\tilde{T}_a(u). \tag{2.36}$$

Introduce the *double-row monodromy matrix*

$$\check{\mathcal{T}}_a(u) = T_a(u)\check{K}_a^-(u)\tilde{T}_a(u). \tag{2.37}$$

Using the relations (2.8), (2.35), (2.36) and the reflection equation (2.32a) one can check that the monodromy matrix given by (2.37) satisfies the following equation in $V_a \otimes V_b \otimes \mathcal{H}$:

$$R_{ab}(u-v)\check{\mathcal{T}}_a(u)R_{ba}(u+v)\check{\mathcal{T}}_b(v) = \check{\mathcal{T}}_b(v)R_{ab}(u+v)\check{\mathcal{T}}_a(u)R_{ba}(u-v). \tag{2.38}$$

Now, the *double-row transfer matrix* is defined as

$$\check{t}(u) = \operatorname{tr}_a\left(\check{K}_a^+(u)\check{\mathcal{T}}_a(u)\right). \tag{2.39}$$

Using (2.38) and the dual reflection equation (2.32b) one can prove that, like in the periodic case, the transfer matrices given by (2.39) commute for any two values of the spectral parameter:

$$[\check{t}(u), \check{t}(v)] = 0 \quad \text{for all} \quad u, v \in \mathbb{C}.$$

Thus, also in this case the transfer matrix can be used it as a generating function for the conserved operators.

The following K-matrix[3] satisfies the reflection equation (2.32a) together with the trigonometric R-matrix (2.2):

$$\check{K}^-(u) = \begin{pmatrix} \sinh(\xi^- + u) & 0 \\ 0 & \sinh(\xi^- - u) \end{pmatrix}. \tag{2.40}$$

[3]In this chapter we only consider the diagonal solutions to the reflection equations (2.32). In the following chapters we will also consider off-diagonal solutions.

Then,

$$\check{K}^{+}(u) = -\left.\check{K}^{-}(-u-\eta)\right|_{\xi^{-}\mapsto -\xi^{+}} = \begin{pmatrix} \sinh(\xi^{+}+u+\eta) & 0 \\ 0 & \sinh(\xi^{+}-u-\eta) \end{pmatrix} \tag{2.41}$$

automatically satisfies the dual reflection equation (2.32b). For subsequent calculations it is convenient to make a variable change $u \mapsto u - \eta/2$, $\varepsilon_l \mapsto \varepsilon_l - \eta/2$ and redefine all functions taking this into account. For the K-matrices (2.40), (2.41) this results in

$$K^{-}(u) = \check{K}^{-}(u-\eta/2) = \begin{pmatrix} \sinh(\xi^{-}+u-\eta/2) & 0 \\ 0 & \sinh(\xi^{-}-u+\eta/2) \end{pmatrix}, \tag{2.42a}$$

$$K^{+}(u) = \check{K}^{+}(u-\eta/2) = \begin{pmatrix} \sinh(\xi^{+}+u+\eta/2) & 0 \\ 0 & \sinh(\xi^{+}-u-\eta/2) \end{pmatrix}. \tag{2.42b}$$

The double-row monodromy matrix (2.37) is now given by

$$\mathcal{T}_a(u) = L_{a\mathcal{L}}(u-\varepsilon_{\mathcal{L}}) \cdots L_{a1}(u-\varepsilon_1) K_a^{-}(u) L_{a1}(u+\varepsilon_1) \cdots L_{a\mathcal{L}}(u+\varepsilon_{\mathcal{L}}), \tag{2.43}$$

and the transfer matrix is, correspondingly,

$$t(u) = \operatorname{tr}_a \left(K_a^{+}(u) L_{a\mathcal{L}}(u-\varepsilon_{\mathcal{L}}) \cdots L_{a1}(u-\varepsilon_1) K_a^{-}(u) L_{a1}(u+\varepsilon_1) \cdots L_{a\mathcal{L}}(u+\varepsilon_{\mathcal{L}}) \right). \tag{2.44}$$

Like in the periodic case, one can write the monodromy matrix (2.43) as an operator valued 2×2-matrix in the auxiliary space:

$$\mathcal{T}_a(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}.$$

Remark 2.8. *Note that we use the same notation for the entries of the monodromy matrix as in the twisted-periodic case. We will keep recycling this notation in the future.*

It is convenient to work with $\tilde{A}(u) = \sinh(2u)A(u) - \sinh\eta D(u)$ instead of $A(u)$. Using

(2.38), one can show that the following commutation relations hold:

$$
\begin{aligned}
D(u)C(v) = {} & \frac{\sinh(u-v-\eta)\sinh(u+v-\eta)}{\sinh(u-v)\sinh(u+v)}\,C(v)D(u) + \\
& + \frac{\sinh\eta\sinh(2v-\eta)}{\sinh(u-v)\sinh(2v)}\,C(u)D(v) - \frac{\sinh\eta}{\sinh(u+v)\sinh(2v)}\,C(u)\tilde{A}(v), \\
\tilde{A}(u)C(v) = {} & \frac{\sinh(u-v+\eta)\sinh(u+v+\eta)}{\sinh(u-v)\sinh(u+v)}\,C(v)\tilde{A}(u) - \\
& - \frac{\sinh\eta\sinh(2u+\eta)}{\sinh(u-v)\sinh(2v)}\,C(u)\tilde{A}(v) + \\
& + \frac{\sinh\eta\sinh(2v-\eta)\sinh(2u+\eta)}{\sinh(u+v)\sinh(2v)}\,C(u)D(v).
\end{aligned}
\tag{2.45}
$$

The transfer matrix (2.44) can be written in the form

$$
t(u) = \frac{\sinh(\xi^+ + u + \eta/2)}{\sinh(2u)}\,\tilde{A}(u) + \frac{\sinh(2u+\eta)\sinh(\xi^+ - u + \eta/2)}{\sinh(2u)}\,D(u).
$$

To find its eigenstates and eigenvalues we follow a generalisation of the algebraic Bethe Ansatz as described in [Skl88]. As in the periodic case, we start with a reference state $\Omega \in V^{\otimes \mathcal{L}}$ satisfying (2.12) and look for other eigenstates in the form (same as (2.13) in the twisted-periodic case)

$$
\Phi = \Phi(v_1, \ldots, v_N) = C(v_1)\cdots C(v_N)\Omega. \tag{2.46}
$$

By linearity from (2.12) we have $\tilde{A}(u)\Omega = \tilde{a}(u)\Omega$, where $\tilde{a}(u) = \sinh(2u)a(u) - \sinh\eta d(u)$. Using relations (2.45) one can prove that the state Φ given by (2.46) is an eigenstate of $t(u)$ with the eigenvalue

$$
\begin{aligned}
\Lambda(u, v_1, \ldots, v_N) = {} & \tilde{a}(u)\frac{\sinh(\xi^+ + u + \eta/2)}{\sinh(2u)} \times \\
& \times \prod_{k=1}^{N} \frac{\sinh(u-v_k+\eta)\sinh(u+v_k+\eta)}{\sinh(u-v_k)\sinh(u+v_k)} + \\
& + d(u)\frac{\sinh(2u+\eta)\sinh(\xi^+ - u + \eta/2)}{\sinh(2u)} \times \\
& \times \prod_{k=1}^{N} \frac{\sinh(u-v_k-\eta)\sinh(u+v_k-\eta)}{\sinh(u-v_k)\sinh(u+v_k)},
\end{aligned}
\tag{2.47}
$$

if and only if $\Phi \neq 0$ and the following BAE are satisfied:

$$\begin{aligned}\frac{\tilde{a}(v_k)}{d(v_k)\sinh(2v_k-\eta)}\frac{\sinh(\xi^+ + v_k+\eta/2)}{\sinh(\xi^+ - v_k+\eta/2)} &= \\ = \prod_{i\neq k}^{N}\frac{\sinh(v_k-v_i-\eta)\sinh(v_k+v_i-\eta)}{\sinh(v_k-v_i+\eta)\sinh(v_k+v_i+\eta)}, &\quad k=1,\ldots,N.\end{aligned} \tag{2.48}$$

Again, one can check that $\Omega = \begin{pmatrix}0\\1\end{pmatrix}^{\otimes\mathcal{L}}$ is a reference state and

$$L_{al}(u-\varepsilon_l)\begin{pmatrix}0\\1\end{pmatrix}_l = \frac{1}{\sinh(u-\varepsilon_l)}\begin{pmatrix}\sinh(u-\varepsilon_l-\eta/2) & 0\\ * & \sinh(u-\varepsilon_l+\eta/2)\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}_l,$$

$$L_{al}(u+\varepsilon_l)\begin{pmatrix}0\\1\end{pmatrix}_l = \frac{1}{\sinh(u+\varepsilon_l)}\begin{pmatrix}\sinh(u+\varepsilon_l-\eta/2) & 0\\ * & \sinh(u+\varepsilon_l+\eta/2)\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}_l.$$

From here one can derive the formulae for $\tilde{a}(u)$ and $d(u)$:

$$\begin{aligned}\tilde{a}(u) &= \sinh(2u-\eta)\sinh(\xi^- + u+\eta/2)\times\\ &\times\prod_{l=1}^{\mathcal{L}}\frac{\sinh(u-\varepsilon_l-\eta/2)\sinh(u+\varepsilon_l-\eta/2)}{\sinh(u-\varepsilon_l)\sinh(u+\varepsilon_l)},\\ d(u) &= \sinh(\xi^- - u+\eta/2)\prod_{l=1}^{\mathcal{L}}\frac{\sinh(u-\varepsilon_l+\eta/2)\sinh(u+\varepsilon_l+\eta/2)}{\sinh(u-\varepsilon_l)\sinh(u+\varepsilon_l)}.\end{aligned} \tag{2.49}$$

2.5 Generalised BQISM

Here we will implement a modification of Sklyanin's formulation, following Karowski and Zapletal [KZ94]. This consists of introducing an additional parameter ρ, which provides a shift in the parameters:

$$u \mapsto u+\frac{\rho}{2},\;\; \varepsilon_l \mapsto \varepsilon_l+\frac{\rho}{2}. \tag{2.50}$$

It will allow us to interpolate between the boundary and the twisted-periodic cases. The limit $\rho \to 0$ reduces to Sklyanin's original boundary formulation, while the limit $\rho \to \infty$, as we will see later, yields the twisted-periodic construction. After implementing the variable change (2.50) the transfer matrix (2.44) becomes

$$
\begin{aligned}
t(u) = \operatorname{tr}_a \Big(K_a^+(u+\rho/2) L_{a\mathcal{L}}(u-\varepsilon_{\mathcal{L}}) \cdots L_{a1}(u-\varepsilon_1) \times \\
\times K_a^-(u+\rho/2) L_{a1}(u+\varepsilon_1+\rho) \cdots L_{a\mathcal{L}}(u+\varepsilon_{\mathcal{L}}+\rho) \Big).
\end{aligned} \tag{2.51}
$$

The eigenvalues (2.47) and the BAE (2.48) are now given by

$$
\begin{aligned}
\Lambda(u, v_1, \ldots, v_N) = \tilde{a}(u) \frac{\sinh(\xi^+ + u + \rho/2 + \eta/2)}{\sinh(2u+\rho)} \times \\
\times \prod_{k=1}^{N} \frac{\sinh(u - v_k + \eta) \sinh(u + v_k + \rho + \eta)}{\sinh(u - v_k) \sinh(u + v_k + \rho)} + \\
+ d(u) \frac{\sinh(2u + \rho + \eta) \sinh(\xi^+ - u - \rho/2 + \eta/2)}{\sinh(2u+\rho)} \times \\
\times \prod_{k=1}^{N} \frac{\sinh(u - v_k - \eta) \sinh(u + v_k + \rho - \eta)}{\sinh(u - v_k) \sinh(u + v_k + \rho)},
\end{aligned} \tag{2.52}
$$

$$
\begin{aligned}
\frac{\tilde{a}(v_k)}{d(v_k) \sinh(2v_k + \rho - \eta)} \frac{\sinh(\xi^+ + v_k + \rho/2 + \eta/2)}{\sinh(\xi^+ - v_k - \rho/2 + \eta/2)} = \\
= \prod_{i \neq k}^{N} \frac{\sinh(v_k - v_i - \eta) \sinh(v_k + v_i + \rho - \eta)}{\sinh(v_k - v_i + \eta) \sinh(v_k + v_i + \rho + \eta)}, \quad k = 1, \ldots, N,
\end{aligned} \tag{2.53}
$$

where

$$
\begin{aligned}
\tilde{a}(u) = \sinh(2u + \rho - \eta) \sinh(\xi^- + u + \rho/2 + \eta/2) \times \\
\times \prod_{l=1}^{\mathcal{L}} \frac{\sinh(u - \varepsilon_l - \eta/2) \sinh(u + \varepsilon_l + \rho - \eta/2)}{\sinh(u - \varepsilon_l) \sinh(u + \varepsilon_l + \rho)}, \\
d(u) = \sinh(\xi^- - u - \rho/2 + \eta/2) \prod_{l=1}^{\mathcal{L}} \frac{\sinh(u - \varepsilon_l + \eta/2) \sinh(u + \varepsilon_l + \rho + \eta/2)}{\sinh(u - \varepsilon_l) \sinh(u + \varepsilon_l + \rho)}.
\end{aligned} \tag{2.54}
$$

Remark 2.9. *In the following we will refer to the shift (2.50) as the variable change #1. According to the algebraic Bethe Ansatz procedure, for the BAE it will result in the change of variables*

$$
v_k \mapsto v_k + \frac{\rho}{2}, \;\; \varepsilon_l \mapsto \varepsilon_l + \frac{\rho}{2}
$$

and for the eigenvalues, correspondingly,

$$
u \mapsto u + \frac{\rho}{2}, \;\; v_k \mapsto v_k + \frac{\rho}{2}, \;\; \varepsilon_l \mapsto \varepsilon_l + \frac{\rho}{2}.
$$

Setting $\rho = 0$ above, the construction reduces to the regular form of the BQISM with inhomogeneities. Below we show that the limit $\rho \to \infty$ reduces to the twisted-periodic QISM formulation, where the twist is sector dependent. We refer to this limit as the *attenuated limit*, since the double row transfer matrix reduces to a single row transfer matrix as $\rho \to \infty$. This approach was used in [KZ94] to construct twisted-periodic one-dimensional quantum lattice models in a manner which preserved certain Hopf-algebraic symmetries.

Remark 2.10. *In the following, we will take various limits of quantities such as the operators $K^{\pm}(u)$ and $L(u)$, the transfer matrix, its eigenvalues and the BAE. For readability we have chosen not to introduce new notation for each limiting object, but will ensure that it is clear which expression is being affected. We will also omit $v_1, \ldots, v_N$ in the notation of the eigenvalues (2.47) and (2.52) writing simply $\Lambda(u)$.*

2.5.1 Attenuated limit

For the Lax operator (2.17) we have

$$L(u) \xrightarrow{u \to \infty} M = \begin{pmatrix} q^{1/2} & 0 & 0 & 0 \\ 0 & q^{-1/2} & 0 & 0 \\ 0 & 0 & q^{-1/2} & 0 \\ 0 & 0 & 0 & q^{1/2} \end{pmatrix},$$

where $q = e^{\eta}$.

Consider a matrix $\hat{N}_l = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_l$ acting on the lth space V_l from the tensor product (2.5). We then have

$$\begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}_l = q^{\hat{N}_l - 1/2}, \qquad \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix}_l = q^{1/2 - \hat{N}_l}.$$

Thus,

$$L_{al}(u) \xrightarrow{u \to \infty} M_l = \begin{pmatrix} q^{\hat{N}_l - 1/2} & 0 \\ 0 & q^{1/2 - \hat{N}_l} \end{pmatrix},$$

and

$$L_{a1}(u + \varepsilon_1 + \rho) \cdots L_{a\mathcal{L}}(u + \varepsilon_{\mathcal{L}} + \rho) \xrightarrow{\rho \to \infty} M_1 M_2 \cdots M_{\mathcal{L}} =$$

$$= \begin{pmatrix} q^{\hat{N}_1 - 1/2} & 0 \\ 0 & q^{1/2 - \hat{N}_1} \end{pmatrix} \cdots \begin{pmatrix} q^{\hat{N}_{\mathcal{L}} - 1/2} & 0 \\ 0 & q^{1/2 - \hat{N}_{\mathcal{L}}} \end{pmatrix} =$$
$$= \begin{pmatrix} q^{\hat{N} - \mathcal{L}/2} & 0 \\ 0 & q^{\mathcal{L}/2 - \hat{N}} \end{pmatrix},$$

where $\hat{N} = \sum_{l=1}^{\mathcal{L}} \hat{N}_l$. A transfer matrix eigenstate Φ is also an eigenstate of the operator $\hat{N}$ with eigenvalue equal to N, the number of C-operators applied to the reference state in order to obtain $\Phi = C(v_1) \cdots C(v_N)\Omega$. In this manner it is seen that the transfer matrix has a block diagonal structure whereby $\hat{N}$ takes a constant value on each block.

Introducing an additional scaling factor $(\sinh u)^{-1}$ into the K-matrices (2.42) and taking the limit as $u \to \infty$ we obtain

$$K^-(u) = \frac{1}{\sinh u} \begin{pmatrix} \sinh(\xi^- + u - \eta/2) & 0 \\ 0 & \sinh(\xi^- - u + \eta/2) \end{pmatrix} \xrightarrow{u\to\infty} \begin{pmatrix} e^{\xi^- - \eta/2} & 0 \\ 0 & -e^{-\xi^- - \eta/2} \end{pmatrix},$$
$$K^+(u) = \frac{1}{\sinh u} \begin{pmatrix} \sinh(\xi^+ + u + \eta/2) & 0 \\ 0 & \sinh(\xi^+ - u - \eta/2) \end{pmatrix} \xrightarrow{u\to\infty} \begin{pmatrix} e^{\xi^+ + \eta/2} & 0 \\ 0 & -e^{-\xi^+ + \eta/2} \end{pmatrix}.$$

Denote

$$L_{a\mathcal{L}}(u - \varepsilon_{\mathcal{L}}) \cdots L_{a1}(u - \varepsilon_1) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}.$$

We then have

$$t(u) \xrightarrow{\rho\to\infty} \mathrm{tr}_a \left(\begin{pmatrix} e^{\xi^+ + \eta/2} & 0 \\ 0 & -e^{-\xi^+ + \eta/2} \end{pmatrix}_a \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \begin{pmatrix} e^{\xi^- - \eta/2} & 0 \\ 0 & -e^{-\xi^- - \eta/2} \end{pmatrix}_a \times \right.$$
$$\left. \times \begin{pmatrix} q^{\hat{N} - \mathcal{L}/2} & 0 \\ 0 & q^{\mathcal{L}/2 - \hat{N}} \end{pmatrix} \right) =$$
$$= \exp(\xi^+ + \xi^-) A(u) \exp\left(\eta(\hat{N} - \mathcal{L}/2)\right) +$$
$$+ \exp(-\xi^+ - \xi^-) D(u) \exp\left(\eta(\mathcal{L}/2 - \hat{N})\right).$$

Since $\hat{N}$ is a conserved operator, it commutes with both $A(u)$ and $D(u)$. Thus,

$$t(u) \xrightarrow{\rho\to\infty} \exp(\xi^+ + \xi^- + \eta N - \eta\mathcal{L}/2) A(u) + \exp(-\xi^+ - \xi^- + \eta\mathcal{L}/2 - \eta N) D(u). \quad (2.55)$$

Remark 2.11. *Recall from Section 2.1 that twisted-periodic transfer matrix (2.9) has the* *form*

$$t(u) = \mathrm{tr}_a \left(\begin{pmatrix} e^{-\eta\gamma} & 0 \\ 0 & e^{\eta\gamma} \end{pmatrix}_a L_{a\mathcal{L}}(u-\varepsilon_{\mathcal{L}}) \cdots L_{a1}(u-\varepsilon_1) \right) = \exp(-\eta\gamma)A(u) + \exp(\eta\gamma)D(u).$$

Thus, to obtain the twisted-periodic transfer matrix (2.9) from the attenuated limit (2.55) of the boundary transfer matrix (2.44), we need to impose that γ depends on N:

$$\gamma = \mathcal{L}/2 - N - \eta^{-1}(\xi^+ + \xi^-). \tag{2.56}$$

From (2.54) we can compute that

$$\frac{\tilde{a}(v_k)}{d(v_k)\sinh(2v_k+\rho-\eta)} = \frac{\sinh(\xi^- + v_k + \rho/2 + \eta/2)}{\sinh(\xi^- - v_k - \rho/2 + \eta/2)} \times \\ \times \prod_{l=1}^{\mathcal{L}} \frac{\sinh(v_k - \varepsilon_l - \eta/2)\sinh(v_k + \varepsilon_l + \rho - \eta/2)}{\sinh(v_k - \varepsilon_l + \eta/2)\sinh(v_k + \varepsilon_l + \rho + \eta/2)}.$$

In the limit as $\rho \to \infty$:

$$\begin{aligned} \frac{\sinh(\xi^- + v_k + \rho/2 + \eta/2)}{\sinh(\xi^- - v_k - \rho/2 + \eta/2)} &\xrightarrow{\rho\to\infty} -\exp(2\xi^- + \eta), \\ \frac{\sinh(\xi^+ + v_k + \rho/2 + \eta/2)}{\sinh(\xi^+ - v_k - \rho/2 + \eta/2)} &\xrightarrow{\rho\to\infty} -\exp(2\xi^+ + \eta), \\ \frac{\sinh(v_k + \varepsilon_l + \rho - \eta/2)}{\sinh(v_k + \varepsilon_l + \rho + \eta/2)} &\xrightarrow{\rho\to\infty} \exp(-\eta), \\ \frac{\sinh(v_k + v_j + \rho - \eta)}{\sinh(v_k + v_j + \rho + \eta)} &\xrightarrow{\rho\to\infty} \exp(-2\eta). \end{aligned}$$

Thus, the BAE (2.53) in this limit reduce to

$$\exp\left(2(\xi^+ + \xi^-) - \eta\mathcal{L} + 2\eta N\right) \prod_{l=1}^{\mathcal{L}} \frac{\sinh(v_k - \varepsilon_l - \eta/2)}{\sinh(v_k - \varepsilon_l + \eta/2)} = \prod_{i\neq k}^{N} \frac{\sinh(v_k - v_i - \eta)}{\sinh(v_k - v_i + \eta)}.$$

We recognise that subject to (2.56) these are equivalent to the BAE (2.21) for (2.9), as required.

In a similar manner we obtain the limit of the eigenvalues (2.52) (taking into account scaling factors in the K-matrices) as

$$\Lambda(u) \xrightarrow{\rho\to\infty} \exp(\xi^+ + \xi^- - \eta\mathcal{L}/2 + \eta N) \prod_{l=1}^{\mathcal{L}} \frac{\sinh(u-\varepsilon_l-\eta/2)}{\sinh(u-\varepsilon_l)} \prod_{k=1}^{N} \frac{\sinh(u-v_k+\eta)}{\sinh(u-v_k)} + \\ + \exp(-\xi^+ - \xi^- + \eta\mathcal{L}/2 - \eta N) \prod_{l=1}^{\mathcal{L}} \frac{\sinh(u-\varepsilon_l+\eta/2)}{\sinh(u-\varepsilon_l)} \prod_{k=1}^{N} \frac{\sinh(u-v_k-\eta)}{\sinh(u-v_k)},$$

which, subject to (2.56), are equal to (2.20) from the twisted-periodic construction.

2.5.2 Rational limit

In this section we show that there is a relationship between the rational twisted-periodic system and the rational boundary system that is similar to the trigonometric case that we have just discussed in the previous section. As discussed before, in the rational limit the Lax operator (2.17) turns into (2.18). In the rational limit from the K-matrices (2.42) with an additional scaling factor u^{-1} we obtain

$$K^-(u) \to \frac{1}{u} \begin{pmatrix} \xi^- + u - \eta/2 & 0 \\ 0 & \xi^- - u + \eta/2 \end{pmatrix}, \tag{2.57a}$$

$$K^+(u) \to \frac{1}{u} \begin{pmatrix} \xi^+ + u + \eta/2 & 0 \\ 0 & \xi^+ - u - \eta/2 \end{pmatrix}. \tag{2.57b}$$

We observe that in this limit, the BAE (2.53) become

$$\frac{(\xi^- + v_k + \rho/2 + \eta/2)(\xi^+ + v_k + \rho/2 + \eta/2)}{(\xi^- - v_k - \rho/2 + \eta/2)(\xi^+ - v_k - \rho/2 + \eta/2)} \times \\ \times \prod_{l=1}^{\mathcal{L}} \frac{(v_k - \varepsilon_l - \eta/2)(v_k + \varepsilon_l + \rho - \eta/2)}{(v_k - \varepsilon_l + \eta/2)(v_k + \varepsilon_l + \rho + \eta/2)} = \\ = \prod_{i\neq k}^{N} \frac{(v_k - v_i - \eta)(v_k + v_i + \rho - \eta)}{(v_k - v_i + \eta)(v_k + v_i + \rho + \eta)}, \tag{2.58}$$

and the expression for the eigenvalues given in (2.52) (taking into account the new scaling factor $(u+\rho/2)^{-1}$ in the K-matrices) can be explicitly written as

$$\begin{aligned}\Lambda(u) \to\, &\frac{(2u+\rho-\eta)}{(2u+\rho)}\frac{(\xi^-+u+\rho/2+\eta/2)(\xi^++u+\rho/2+\eta/2)}{(u+\rho/2)^2}\times\\ &\times\prod_{l=1}^{\mathcal{L}}\frac{(u-\varepsilon_l-\eta/2)(u+\varepsilon_l+\rho-\eta/2)}{(u-\varepsilon_l)(u+\varepsilon_l+\rho)}\prod_{k=1}^{N}\frac{(u-v_k+\eta)(u+v_k+\rho+\eta)}{(u-v_k)(u+v_k+\rho)}+\\ &+\frac{(2u+\rho+\eta)}{(2u+\rho)}\frac{(\xi^--u-\rho/2+\eta/2)(\xi^+-u-\rho/2+\eta/2)}{(u+\rho/2)^2}\times\\ &\times\prod_{l=1}^{\mathcal{L}}\frac{(u-\varepsilon_l+\eta/2)(u+\varepsilon_l+\rho+\eta/2)}{(u-\varepsilon_l)(u+\varepsilon_l+\rho)}\prod_{k=1}^{N}\frac{(u-v_k-\eta)(u+v_k+\rho-\eta)}{(u-v_k)(u+v_k+\rho)}.\end{aligned}\tag{2.59}$$

The transfer matrix (2.51) in the rational limit is readily obtained by employing the expressions (2.18) and (2.57). To determine the attenuated limit of this rational transfer matrix, we first observe that from (2.18), $L(u)\to I$ as $u\to\infty$. This implies that the terms $L_{aj}(u+\varepsilon_j+\rho)$ occurring to the right of $K_a^-(u+\rho/2)$ in (2.51) all simplify to the identity as $\rho\to\infty$. Without loss of generality, we moreover suppose that ξ^- does not depend on ρ, in which case taking the attenuated limit of (2.57a) gives

$$K^-(u+\rho/2)\xrightarrow{\rho\to\infty}\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

Furthermore, we set $\xi^+=\zeta\rho$, where $\zeta\in\mathbb{C}$, from which we obtain the attenuated limit of equation (2.57b) above:

$$K^+(u+\rho/2)\xrightarrow{\rho\to\infty}\begin{pmatrix}2\zeta+1&0\\0&2\zeta-1\end{pmatrix}.$$

Thus, we have the attenuated limit of the rational transfer matrix in the form (2.51) being given by

$$\begin{aligned}t(u)\xrightarrow{\rho\to\infty}\operatorname{tr}_a&\left(\begin{pmatrix}2\zeta+1&0\\0&2\zeta-1\end{pmatrix}_a L_{a\mathcal{L}}(u-\varepsilon_{\mathcal{L}})\cdots L_{a1}(u-\varepsilon_1)\begin{pmatrix}1&0\\0&-1\end{pmatrix}_a\right)=\\ &=(1+2\zeta)A(u)+(1-2\zeta)D(u),\end{aligned}$$

where the operators $L_{aj}(u-\varepsilon_j)$ and, correspondingly, the operators $A(u)$ and $D(u)$ are in the rational limit.

Finally, imposing the condition that $\zeta\neq\pm1/2$ to avoid any technical issues of divergence, for convenience we rescale

$$K^+(u+\rho/2) \mapsto \frac{1}{\sqrt{1-4\zeta^2}} K^+(u+\rho/2)$$

to match this limiting expression for $t(u)$ with the rational limit of the twisted-periodic transfer matrix given in equation (2.9) above. This is achieved by setting

$$e^{-\eta\gamma} = \frac{1+2\zeta}{\sqrt{1-4\zeta^2}}, \quad e^{\eta\gamma} = \frac{1-2\zeta}{\sqrt{1-4\zeta^2}}. \tag{2.60}$$

In the attenuated limit (i.e., $\rho \to \infty$), the rational BAE (2.58) become

$$\frac{1+2\zeta}{1-2\zeta} \prod_{l=1}^{\mathcal{L}} \frac{v_k - \varepsilon_l - \eta/2}{v_k - \varepsilon_l + \eta/2} = \prod_{i \neq k}^{N} \frac{v_k - v_i - \eta}{v_k - v_i + \eta}. \tag{2.61}$$

It is evident that, taking (2.60) into account, we may identify (2.61) with the rational limit of (2.21).

Finally, the expression for the eigenvalues (2.59) in the attenuated limit is

$$\begin{aligned} \Lambda(u) \to \frac{1+2\zeta}{\sqrt{1-4\zeta^2}} \prod_{l=1}^{\mathcal{L}} \frac{u-\varepsilon_l-\eta/2}{u-\varepsilon_l} \prod_{k=1}^{N} \frac{u-v_k+\eta}{u-v_k} + \\ + \frac{1-2\zeta}{\sqrt{1-4\zeta^2}} \prod_{l=1}^{\mathcal{L}} \frac{u-\varepsilon_l+\eta/2}{u-\varepsilon_l} \prod_{k=1}^{N} \frac{u-v_k-\eta}{u-v_k}. \end{aligned} \tag{2.62}$$

By once again applying (2.60), we may identify the expression (2.62) with the rational limit of (2.20). In other words, we have shown that the rational and attenuated limits *commute*, subject to appropriate scaling of relevant quantities.

A convenient way to summarise our discussions so far in this section is to provide a diagram (Figure 2.1) highlighting the connections we have made between the various trigonometric, hereafter denoted **Trig.**, and rational, hereafter denoted **Rat.**, constructions. We will also use the notations **BQISM** to denote Sklyanin's boundary construction from Section 2.4, and **QISM** for the twisted-periodic construction described in Section 2.3 (which can be obtained in the attenuated limit). **Trig. BQISM′** and **Rat. BQISM′** are merely the respective **Trig. BQISM** and **Rat. BQISM** with ρ included explicitly in all expressions via the variable change #1, denoted simply by #1 in the diagram, which was introduced in in the beginning of the current section (see Remark 2.9). We do not consider these to be fundamentally different systems, but make the distinction as a convenience to highlight our utilisation of the methods of Karowski and Zapletal [KZ94] via the attenuated limit.

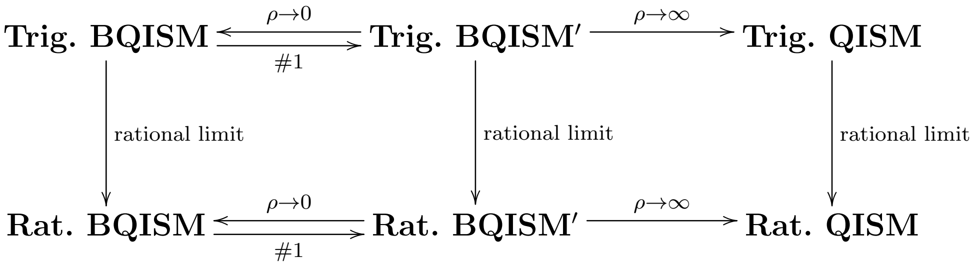

Figure 2.1: Connections between the (diagonal) boundary and twisted-periodic QISM constructions.

2.5.3 Heisenberg model

In this section we show how the Heisenberg model can be obtained as a special case from the general construction outlined so far. Here the inhomogeneity parameters ε_l are set to be zero, the Lax operator $L(u)$ is set to be the R-matrix $R(u)$ itself, and we omit the shift $u \mapsto u - \eta/2$ described in equations (2.42).

Thus, the transfer matrix is

$$t(u) = \mathrm{tr}_a \Big(\check{K}_a^+(u+\rho/2) R_{a\mathcal{L}}(u) \cdots R_{a1}(u) \check{K}_a^-(u+\rho/2) R_{a1}(u+\rho) \cdots R_{a\mathcal{L}}(u+\rho) \Big). \quad (2.63)$$

If we take $\rho \to 0$ we obtain the open chain Heisenberg model transfer matrix:

$$t(u) \to \mathrm{tr}_a \left(\check{K}_a^+(u) R_{a\mathcal{L}}(u) \cdots R_{a1}(u) \check{K}_a^-(u) R_{a1}(u) \cdots R_{a\mathcal{L}}(u) \right). \quad (2.64)$$

The Hamiltonian is constructed from $t(u)$ given by (2.64) as follows:

$$H = t^{-1}(0) t'(0) = \sum_{j=1}^{\mathcal{L}-1} H_{j(j+1)} + \frac{1}{2} \left(\check{K}_1^-(0) \right)^{-1} \left(\check{K}_1^- \right)'(0) + \frac{\mathrm{tr}_a \left(\check{K}_a^+(0) H_{a\mathcal{L}} \right)}{\mathrm{tr}_a \left(\check{K}_a^+(0) \right)}, \quad (2.65)$$

where $H_{j(j+1)} = P_{j(j+1)} R'_{j(j+1)}(0),\ H_{a\mathcal{L}} = R'_{a\mathcal{L}}(0) P_{a\mathcal{L}}$, and $t'(0),\ R'_{j(j+1)}(0)$ and $\left(\check{K}_1^- \right)'(0)$ are derivatives of the corresponding operators at $u = 0$. The explicit form of the Hamiltonian (2.65) in terms of Pauli matrices may be found in [Skl88].

Now if we consider $\rho \to \infty$, the transfer matrix (2.63) will tend to

$$t(u) \to \exp(\xi^+ + \xi^- + \eta N - \eta \mathcal{L}/2) A(u) + \exp(-\xi^+ - \xi^- + \eta \mathcal{L}/2 - \eta N) D(u),$$

where

$$R_{a\mathcal{L}}(u) \cdots R_{a1}(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}.$$

By choosing $\gamma = \mathcal{L}/2 - N - \eta^{-1}(\xi^+ + \xi^-)$ we can match it with the transfer matrix for the closed chain, namely

$$t(u) = \exp(-\eta\gamma) A(u) + \exp(\eta\gamma) D(u) = \mathrm{tr}_a \left(\begin{pmatrix} e^{-\eta\gamma} & 0 \\ 0 & e^{\eta\gamma} \end{pmatrix}_a R_{a\mathcal{L}}(u) \cdots R_{a1}(u) \right).$$

Here again

$$H = t^{-1}(0) t'(0) = \sum_{j=1}^{\mathcal{L}-1} H_{j(j+1)} + X_{\mathcal{L}}^{-1} H_{\mathcal{L}1} X_{\mathcal{L}} = \sum_{j=1}^{\mathcal{L}-1} H_{j(j+1)} + X_1 H_{\mathcal{L}1} X_1^{-1},$$

where $H_{j(j+1)} = P_{j(j+1)} R'_{j(j+1)}(0)$ and $X = \begin{pmatrix} e^{-\eta\gamma} & 0 \\ 0 & e^{\eta\gamma} \end{pmatrix}$.

In the rational limit (XXX model), the calculations are completely analogous to Section 2.5.2, so we omit the details.

As in Section 2.5.2, we may summarise the analogous connections for the Heisenberg model in Figure 2.2. It is worth highlighting the fact that for the Heisenberg case, since we have set the parameters $\varepsilon_j = 0$, it is not possible to implement the variable change #1 discussed in the previous section.

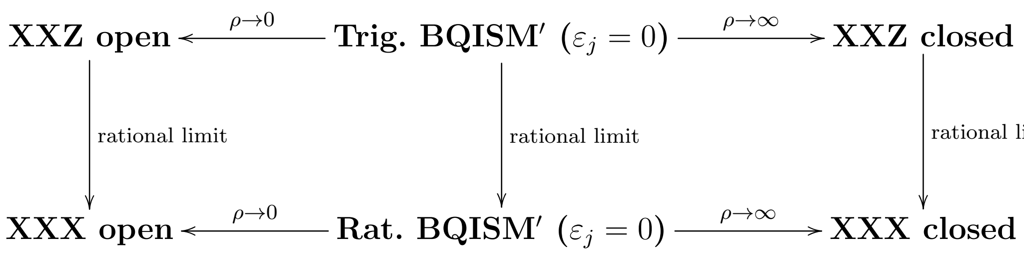

Figure 2.2: Connections between the Heisenberg models (with and without boundary).

Chapter 3

Richardson–Gaudin models with diagonal reflection matrices

In this chapter we investigate the quasi-classical limit of the system described in Section 2.4 (and its generalisation introduced in Section 2.5), which involves expanding all expressions in η as $\eta \to 0$ and taking the first non-trivial term in the expansion.

In the quasi-classical limit, unlike the special case of the Heisenberg model (Section 2.5.3), we are able to implement variable change #1. Moreover, we gain the capability of implementing two additional variable changes. It is through these variable changes that we are able to make unexpected connections between various systems in the quasi-classical limit. We find that the commutative diagram presented in Figure 3.1, in contrast to those presented in Section 2.5, illustrates the connections we shall make in this chapter.

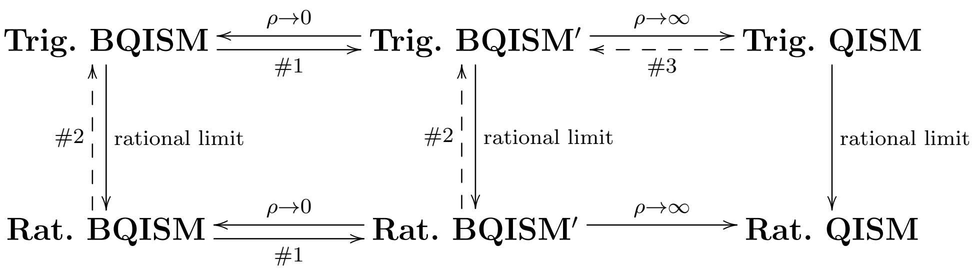

Figure 3.1: Proposed connections between the Richardson–Gaudin models obtained from the BQISM with diagonal reflection matrices.

The connections that have been established previously (see Figure 2.1) still hold in the quasi-classical limit. Dashed arrows represent the connections that are yet to be established. In the diagram we adopt the notation where #1 denotes variable change #1, #2 is used for variable change #2 combined with some other operations, and #3 represents variable change #3 with different operations, all of which are specified explicitly in the text below.

First of all, we are going to establish the above connections for the BAE (Section 3.1) and then we show that the same connections also hold on the level of the conserved operators (Section 3.2) and their eigenvalues (Section 3.3).

3.1 Bethe Ansatz Equations

We start by considering the BAE. Substituting the expressions (2.49) for $\tilde{a}(u)$ and $d(u)$ into the BAE (2.48) gives

$$\begin{aligned}&\frac{\sinh(\xi^+ + v_k + \eta/2)}{\sinh(\xi^+ - v_k + \eta/2)}\frac{\sinh(\xi^- + v_k + \eta/2)}{\sinh(\xi^- - v_k + \eta/2)}\times\\&\times\prod_{l=1}^{\mathcal{L}}\frac{\sinh(v_k - \varepsilon_l - \eta/2)\sinh(v_k + \varepsilon_l - \eta/2)}{\sinh(v_k - \varepsilon_l + \eta/2)\sinh(v_k + \varepsilon_l + \eta/2)} =\\&\qquad = \prod_{i\neq k}^{N}\frac{\sinh(v_k - v_i - \eta)\sinh(v_k + v_i - \eta)}{\sinh(v_k - v_i + \eta)\sinh(v_k + v_i + \eta)}.\end{aligned} \tag{3.1}$$

If we set $\eta = 0$ in (3.1), the expression reduces to

$$\frac{\sinh(\xi^- + v_k)\sinh(\xi^+ + v_k)}{\sinh(\xi^- - v_k)\sinh(\xi^+ - v_k)} = 1. \tag{3.2}$$

Furthermore, we assume that $\xi^\pm$ depend on η in such a way that (3.2) holds as $\eta \to 0$, to ensure that the quasi-classical limit is well-defined. We impose the following choice which is consistent with that property:

$$\xi^+ = \xi + \eta\alpha,\ \xi^- = -\xi + \eta\beta. \tag{3.3}$$

Remark 3.1. *For instance, if we take* $\xi^+ = \xi_1 + \eta\alpha,\ \xi^- = \xi_2 + \eta\beta,\ \xi_1 \neq \xi_2$*, the identity* (3.2) *will not generally hold when we set* $\eta = 0$.

The expansion up to first order in η for the right hand side of the BAE (3.1) with

(3.3) is given by

$$\prod_{j\neq k}^{N} \frac{\big(\sinh(v_k - v_j) - \eta\cosh(v_k - v_j)\big)\big(\sinh(v_k + v_j) - \eta\cosh(v_k + v_j)\big)}{\big(\sinh(v_k - v_j) + \eta\cosh(v_k - v_j)\big)\big(\sinh(v_k + v_j) + \eta\cosh(v_k + v_j)\big)} + \mathcal{O}(\eta^2) =$$
$$= \prod_{j\neq k}^{N} \frac{\big(1 - \eta\coth(v_k - v_j)\big)\big(1 - \eta\coth(v_k + v_j)\big)}{\big(1 + \eta\coth(v_k - v_j)\big)\big(1 + \eta\coth(v_k + v_j)\big)} + \mathcal{O}(\eta^2) =$$
$$= \prod_{j\neq k}^{N} \big(1 - 2\eta\coth(v_k - v_j)\big)\big(1 - 2\eta\coth(v_k + v_j)\big) + \mathcal{O}(\eta^2) =$$
$$= 1 - 2\eta \sum_{j\neq k}^{N} \big(\coth(v_k - v_j) + \coth(v_k + v_j)\big) + \mathcal{O}(\eta^2).$$

Now let us look at the expansion of the left hand side:

$$\prod_{l=1}^{\mathcal{L}} \frac{\sinh(v_k - \varepsilon_l - \eta/2)\sinh(v_k + \varepsilon_l - \eta/2)}{\sinh(v_k - \varepsilon_l + \eta/2)\sinh(v_k + \varepsilon_l + \eta/2)} =$$
$$= 1 - \eta \sum_{l=1}^{\mathcal{L}} \big(\coth(v_k - \varepsilon_l) + \coth(v_k + \varepsilon_l)\big) + \mathcal{O}(\eta^2),$$
$$\frac{\sinh(\xi^+ + v_k + \eta/2)}{\sinh(\xi^+ - v_k + \eta/2)} = \frac{\sinh\left(v_k + \eta(\alpha + 1/2)\right)}{\sinh\left(-v_k + \eta(\alpha + 1/2)\right)} = \frac{1 + \eta(\alpha + 1/2)\coth v_k}{-1 + \eta(\alpha + 1/2)\coth v_k} + \mathcal{O}(\eta^2) =$$
$$= -1 - 2\eta(\alpha + 1/2)\coth v_k + \mathcal{O}(\eta^2),$$
$$\frac{\sinh(\xi^- + v_k + \eta/2)}{\sinh(\xi^- - v_k + \eta/2)} = -1 - 2\eta(\beta + 1/2)\coth v_k + \mathcal{O}(\eta^2).$$

Putting these together we obtain that, up to first order in η, the expansion of the left hand side of (3.1) is

$$1 - \eta(\alpha + \beta + 1)\big(\coth(v_k - \xi) + \coth(v_k + \xi)\big) - \eta \sum_{l=1}^{\mathcal{L}} \big(\coth(v_k - \varepsilon_l) + \coth(v_k + \varepsilon_l)\big).$$

Let us denote $\delta = -(\alpha + \beta + 1)$. Then, in the quasi-classical limit as $\eta \to 0$, the BAE in the case **Trig. BQISM** are given by

$$\begin{aligned} \delta\big(\coth(v_k - \xi) + \coth(v_k + \xi)\big) + \sum_{l=1}^{\mathcal{L}} \big(\coth(v_k - \varepsilon_l) + \coth(v_k + \varepsilon_l)\big) = \\ = 2\sum_{i\neq k}^{N} \big(\coth(v_k - v_i) + \coth(v_k + v_i)\big). \end{aligned} \tag{3.4}$$

3.1.1 Variable change #1

Variable change #1

$$v_k \mapsto v_k + \frac{\rho}{2},\ \varepsilon_l \mapsto \varepsilon_l + \frac{\rho}{2} \tag{3.5}$$

turns **Trig. BQISM** (3.4) into **Trig. BQISM′**:

$$\begin{aligned} &\delta\big(\coth(v_k + \rho/2 - \xi) + \coth(v_k + \rho/2 + \xi)\big) + \\ &+ \sum_{l=1}^{\mathcal{L}} \big(\coth(v_k - \varepsilon_l) + \coth(v_k + \varepsilon_l + \rho)\big) = \\ &\qquad = 2\sum_{i\neq k}^{N} \big(\coth(v_k - v_i) + \coth(v_k + v_i + \rho)\big). \end{aligned} \tag{3.6}$$

It is straightforward to see $\rho \to 0$ turns (3.6) back into (3.4).

3.1.2 Attenuated limit

As $\rho \to \infty$ **Trig. BQISM′** (3.6) reduces to **Trig. QISM** (2.22) in the quasi-classical limit:

$$2\delta + \sum_{l=1}^{\mathcal{L}} \big(\coth(v_k - \varepsilon_l) + 1\big) = 2\sum_{i\neq k}^{N} \big(\coth(v_k - v_i) + 1\big),$$

or

$$2\gamma + \sum_{l=1}^{\mathcal{L}} \coth(v_k - \varepsilon_l) = 2\sum_{i\neq k}^{N} \coth(v_k - v_i),$$

where $\gamma = \delta + \mathcal{L}/2 - (N-1) = -(\alpha + \beta + N - \mathcal{L}/2)$.

3.1.3 Rational limit

Introducing a parameter ν into **Trig. BQISM′** (3.6) we obtain

$$\begin{aligned} &\delta\left(\coth\left(\nu(v_k + \rho/2 - \xi)\right) + \coth\left(\nu(v_k + \rho/2 + \xi)\right)\right) + \\ &+ \sum_{l=1}^{\mathcal{L}} \left(\coth\left(\nu(v_k - \varepsilon_l)\right) + \coth\left(\nu(v_k + \varepsilon_l + \rho)\right)\right) = \\ &\qquad = 2\sum_{i\neq k}^{N} \left(\coth\left(\nu(v_k - v_i)\right) + \coth\left(\nu(v_k + v_i + \rho)\right)\right). \end{aligned}$$

Multiplying by ν we obtain, since $\lim_{\nu\to 0}\big(\nu\coth(\nu x)\big) = \frac{1}{x}$, **Rat. BQISM′** as $\nu \to 0$:

$$\frac{\delta}{(v_k+\rho/2)^2-\xi^2}+\sum_{l=1}^{\mathcal{L}}\frac{1}{(v_k+\rho/2)^2-(\varepsilon_l+\rho/2)^2}=2\sum_{i\neq k}^{N}\frac{1}{(v_k+\rho/2)^2-(v_i+\rho/2)^2}, \quad (3.7)$$

which turns into **Rat. BQISM** as $\rho \to 0$:

$$\frac{\delta}{v_k^2-\xi^2}+\sum_{l=1}^{\mathcal{L}}\frac{1}{v_k^2-\varepsilon_l^2}=2\sum_{i\neq k}^{N}\frac{1}{v_k^2-v_i^2}. \quad (3.8)$$

3.1.4 Rational BQISM and trigonometric QISM equivalence

Making a change of variables $v_k \mapsto \ln y_k,\ \varepsilon_l \mapsto \ln z_l$ in **Trig. QISM** (2.22) we obtain

$$2\gamma+\sum_{l=1}^{\mathcal{L}}\frac{y_k^2+z_l^2}{y_k^2-z_l^2}=2\sum_{i\neq k}^{N}\frac{y_k^2+y_i^2}{y_k^2-y_i^2},$$

which can be rewritten as

$$2\delta+\sum_{l=1}^{\mathcal{L}}\left(\frac{y_k^2+z_l^2}{y_k^2-z_l^2}+1\right)=2\sum_{i\neq k}^{N}\left(\frac{y_k^2+y_i^2}{y_k^2-y_i^2}+1\right)$$

with $\delta=\gamma-\mathcal{L}/2+(N-1)$ as above. Or, after a simplification,

$$\delta+\sum_{l=1}^{\mathcal{L}}\frac{y_k^2}{y_k^2-z_l^2}=2\sum_{i\neq k}^{N}\frac{y_k^2}{y_k^2-y_i^2}. \quad (3.9)$$

Note that (3.9) turns into **Rat. BQISM** (3.8) under the following (invertible) variable change:

$$y_k \mapsto \sqrt{v_k^2-\xi^2},\ z_l \mapsto \sqrt{\varepsilon_l^2-\xi^2}.$$

Combining these variable changes, we obtain that **Trig. QISM** is equivalent to **Rat. BQISM** via the variable change from (2.22) to (3.8) given by

$$v_k \mapsto \ln\sqrt{v_k^2-\xi^2},\ \varepsilon_l \mapsto \ln\sqrt{\varepsilon_l^2-\xi^2}, \quad (3.10)$$

and its inverse

$$v_k \mapsto \sqrt{\exp(2v_k)+\xi^2},\ \varepsilon_l \mapsto \sqrt{\exp(2\varepsilon_l)+\xi^2}$$

which obviously maps from (3.8) to (2.22).

3.1.5 Variable change #2

It can be seen that we may transform from **Rat. BQISM** (3.8) to **Trig. BQISM** (3.4) by a suitable variable change. Application of

$$v_k \mapsto \frac{y_k - y_k^{-1}}{2}, \ \varepsilon_l \mapsto \frac{z_l - z_l^{-1}}{2}, \ \xi \mapsto \frac{\chi - \chi^{-1}}{2}$$

to **Rat. BQISM** (3.8) gives

$$\frac{\delta}{(y_k - y_k^{-1})^2 - (\chi - \chi^{-1})^2} + \sum_{l=1}^{\mathcal{L}} \frac{1}{(y_k - y_k^{-1})^2 - (z_l - z_l^{-1})^2} = \sum_{i \neq k}^{N} \frac{2}{(y_k - y_k^{-1})^2 - (y_i - y_i^{-1})^2}.$$

Multiplying both sides by $y_k^2 - y_k^{-2}$

$$\frac{\delta(y_k^2 - y_k^{-2})}{(y_k - y_k^{-1})^2 - (\chi - \chi^{-1})^2} + \sum_{l=1}^{\mathcal{L}} \frac{y_k^2 - y_k^{-2}}{(y_k - y_k^{-1})^2 - (z_l - z_l^{-1})^2} = \sum_{i \neq k}^{N} \frac{2(y_k^2 - y_k^{-2})}{(y_k^2 + y_k^{-1})^2 - (y_i - y_i^{-1})^2},$$

and simplifying, we obtain

$$\delta \frac{y_k^2 - y_k^{-2}}{y_k^2 + y_k^{-2} - \chi^2 - \chi^{-2}} + \sum_{l=1}^{\mathcal{L}} \frac{y_k^2 - y_k^{-2}}{y_k^2 + y_k^{-2} - z_l^2 - z_l^{-2}} = 2 \sum_{i \neq k}^{N} \frac{y_k^2 - y_k^{-2}}{y_k^2 + y_k^{-2} - y_i^2 - y_i^{-2}}.$$

One can rewrite it as

$$\delta \left(\frac{y_k^2}{y_k^2 - \chi^2} + \frac{y_k^{-2}\chi^{-2}}{1 - y_k^{-2}\chi^{-2}} \right) + \sum_{l=1}^{\mathcal{L}} \left(\frac{y_k^2}{y_k^2 - z_l^2} + \frac{y_k^{-2} z_l^{-2}}{1 - y_k^{-2} z_l^{-2}} \right) =$$
$$= 2 \sum_{i \neq k}^{N} \left(\frac{y_k^2}{y_k^2 - y_i^2} + \frac{y_k^{-2} y_i^{-2}}{1 - y_k^{-2} y_i^{-2}} \right),$$

which is equivalent to

$$\delta \left(\frac{y_k^2}{y_k^2 - \chi^2} + \frac{1}{y_k^2 \chi^2 - 1} \right) + \sum_{l=1}^{\mathcal{L}} \left(\frac{y_k^2}{y_k^2 - z_l^2} + \frac{1}{y_k^2 z_l^2 - 1} \right) = 2 \sum_{i \neq k}^{N} \left(\frac{y_k^2}{y_k^2 - y_i^2} + \frac{1}{y_k^2 y_i^2 - 1} \right),$$

or

$$\delta\left(\frac{y_k^2+\chi^2}{y_k^2-\chi^2}+\frac{y_k^2\chi^2+1}{y_k^2\chi^2-1}\right)+\sum_{l=1}^{\mathcal{L}}\left(\frac{y_k^2+z_l^2}{y_k^2-z_l^2}+\frac{y_k^2z_l^2+1}{y_k^2z_l^2-1}\right)=2\sum_{i\neq k}^{N}\left(\frac{y_k^2+y_i^2}{y_k^2-y_i^2}+\frac{y_k^2y_i^2+1}{y_k^2y_i^2-1}\right).$$

Now, in order to transform this expression into **Trig. BQISM** (3.4) we make a change of variables

$$y_k\mapsto \exp v_k,\;\; z_l\mapsto \exp\varepsilon_l,\;\; \chi\mapsto\exp\xi.$$

Thus, the mapping from **Rat. BQISM** (3.8) to **Trig. BQISM** (3.4) is a composition

$$\begin{aligned} v_k &\mapsto \frac{e^{v_k}-e^{-v_k}}{2}=\sinh v_k,\\ \varepsilon_l &\mapsto \frac{e^{\varepsilon_l}-e^{-\varepsilon_l}}{2}=\sinh\varepsilon_l,\\ \xi &\mapsto \frac{e^{\xi}-e^{-\xi}}{2}=\sinh\xi. \end{aligned} \tag{3.11}$$

Analogously, including ρ gives the mapping from **Rat. BQISM′** (3.7) to **Trig. BQISM′** (3.6):

$$\begin{aligned} v_k+\rho/2 &\mapsto \sinh(v_k+\rho/2),\\ \varepsilon_l+\rho/2 &\mapsto \sinh(\varepsilon_l+\rho/2),\\ \xi &\mapsto \sinh\xi. \end{aligned} \tag{3.12}$$

Generally, we refer to equations (3.12) as the variable change #2, and note that (3.11) is merely a specialisation of (3.12) with $\rho=0$.

3.1.6 Variable change #3

Now, we define the variable change #3 to be a composition of operations defined so far:

Trig. QISM (2.22) $\xrightarrow{(3.10)}$ **Rat. BQISM** (3.8) $\xrightarrow{(3.11)}$ **Trig. BQISM** (3.4) $\xrightarrow{(3.5)}$ **Trig. BQISM′** (3.6).

This results in the variable change given by

$$\begin{aligned} v_k &\mapsto \ln\sqrt{\sinh^2(v_k+\rho/2)-\sinh^2\xi},\\ \varepsilon_l &\mapsto \ln\sqrt{\sinh^2(\varepsilon_l+\rho/2)-\sinh^2\xi}. \end{aligned} \tag{3.13}$$

Equivalently, we may take

Trig. QISM (2.22) $\xrightarrow{(3.10)}$ **Rat. BQISM** (3.8) $\xrightarrow{(3.5)}$ **Rat. BQISM′** (3.7) $\xrightarrow{(3.12)}$ **Trig. BQISM′** (3.6),

which gives the same. We refer to the (3.13) as variable change #3.

3.1.7 Reduction to the rational, twisted-periodic case

Rat. QISM (2.29) is obtained by taking the rational limit of **Trig. QISM** (2.22) as described in Section 2.3.4.

We can also obtain **Rat. QISM** (2.29) by taking the attenuated limit from **Rat. BQISM′** (3.7):

$$\delta + \sum_{l=1}^{\mathcal{L}} \frac{v_k^2 + \rho v_k + \rho^2/4 - \xi^2}{v_k^2 - \varepsilon_l^2 + \rho(v_k - \varepsilon_l)} = 2 \sum_{i \neq k}^{N} \frac{v_k^2 + \rho v_k + \rho^2/4 - \xi^2}{v_k^2 - v_i^2 + \rho(v_k - v_i)}.$$

Rescale the constant $\delta = \rho\gamma/2$, divide throughout by $\rho/4$ and consider $\rho \to \infty$. Then we obtain again **Rat. QISM** (2.29).

We may summarise the connections made in this section in Figure 3.2 below.

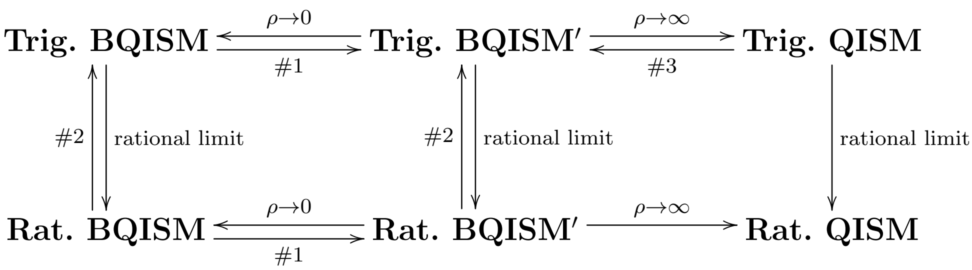

Figure 3.2: Established connections between the Richardson–Gaudin models obtained from the BQISM with diagonal reflection matrices.

It turns out that the limit labelled **Rat. QISM** is not equivalent to any of the other five nodes in the diagram. This is deduced by knowledge of a particular solution of the BAE. For the BAE (3.9), it was identified in [ILSZ09] that when $\delta = N - 1$ there is a solution for which $y_k = 0$ for all k. Results from [RDO10] show that such a solution where all roots are equal does not exist for the BAE (2.29). Consequently (3.9) and (2.29) cannot be equivalent.

The most unexpected aspect of the above calculations concerns the parameter ξ. Recall that this parameter arises in the expansion of the variables $\xi^{\pm}$, as given by (3.3), where $\xi^{\pm}$ are the free parametrising variables of the reflection matrices (2.42). The above calculations show that ξ is a spurious variable which can be removed by appropriate variable changes. In the next section we will show that it is also possible to remove the ξ-dependence from the conserved operators, but this requires an appropriate rescaling and basis transformation in conjunction with the variable changes.

3.2 Conserved operators

3.2.1 The first family of conserved operators

Similarly to Section 2.3.2, the conserved operators τ_j in the quasi-classical limit are constructed as follows from the transfer matrix (2.44):

$$\lim_{u \to \varepsilon_j} (u - \varepsilon_j) t(u) = \eta^2 \tau_j + \mathcal{O}(\eta^3). \tag{3.14}$$

To calculate these conserved operators, we impose the conditions (3.3) on parameters $\xi^{\pm}$ that appear in the reflection matrices (2.42). Note that (3.3) implies that

$$\lim_{\eta \to 0} \{K^+(u) K^-(u)\} \propto I.$$

This ensures that the transfer matrix (2.44) satisfies $\lim_{\eta \to 0} t(u) \propto I$, which allows the quasi-classical expansion of the transfer matrix to obtain conserved operators.

Expanding $K^{\pm}(u)$ in powers of η as $\eta \to 0$ then gives

$$K^+(u) = K_1^+(u) + \eta K_2^+(u) + \mathcal{O}(\eta^2), \quad K^-(u) = K_1^-(u) + \eta K_2^-(u) + \mathcal{O}(\eta^2), \tag{3.15}$$

where we define

$$K_1^+(u) = \begin{pmatrix} \sinh(\xi + u) & 0 \\ 0 & \sinh(\xi - u) \end{pmatrix},$$

$$K_2^+(u) = \begin{pmatrix} \left(\alpha+\frac{1}{2}\right)\cosh(\xi+u) & 0 \\ 0 & \left(\alpha-\frac{1}{2}\right)\cosh(\xi-u) \end{pmatrix},$$
$$K_1^-(u) = -\begin{pmatrix} \sinh(\xi-u) & 0 \\ 0 & \sinh(\xi+u) \end{pmatrix},$$
$$K_2^-(u) = \begin{pmatrix} \left(\beta-\frac{1}{2}\right)\cosh(\xi-u) & 0 \\ 0 & \left(\beta+\frac{1}{2}\right)\cosh(\xi+u) \end{pmatrix}.$$

Using the expressions of equations (3.15) above and the quasi-classical expansion of $L_{al}(u)$ (2.24), we may expand (3.14) explicitly as

$$\begin{aligned}
&\lim_{u\to\varepsilon_j}(u-\varepsilon_j)t(u) = \\
&= \eta\,\mathrm{tr}_a\Bigg[K_{1a}^+(\varepsilon_j)\ell_{aj}(0)K_{1a}^-(\varepsilon_j) + \eta\sum_{k=j+1}^{\mathcal{L}}\frac{K_{1a}^+(\varepsilon_j)\ell_{ak}(\varepsilon_j-\varepsilon_k)\ell_{aj}(0)K_{1a}^-(\varepsilon_j)}{\sinh(\varepsilon_j-\varepsilon_k)} + \\
&\qquad + \eta\sum_{k=1}^{j-1}\frac{K_{1a}^+(\varepsilon_j)\ell_{aj}(0)\ell_{ak}(\varepsilon_j-\varepsilon_k)K_{1a}^-(\varepsilon_j)}{\sinh(\varepsilon_j-\varepsilon_k)} + \eta K_{2a}^+(\varepsilon_j)\ell_{aj}(0)K_{1a}^-(\varepsilon_j) + \\
&\qquad + \eta\sum_{k=1}^{\mathcal{L}}\frac{K_{1a}^+(\varepsilon_j)\ell_{aj}(0)K_{1a}^-(\varepsilon_j)\ell_{ak}(\varepsilon_j+\varepsilon_k)}{\sinh(\varepsilon_j+\varepsilon_k)} + \eta K_{1a}^+(\varepsilon_j)\ell_{aj}(0)K_{2a}^-(\varepsilon_j)\Bigg] + \mathcal{O}(\eta^3).
\end{aligned}$$

One can check that

$$\mathrm{tr}_a\left(K_{1a}^+(\varepsilon_j)\ell_{aj}(0)K_{1a}^-(\varepsilon_j)\right) = 0,$$
$$\mathrm{tr}_a\left(K_{1a}^+(\varepsilon_j)\ell_{ak}(\varepsilon_j-\varepsilon_k)\ell_{aj}(0)K_{1a}^-(\varepsilon_j)\right) = \mathrm{tr}_a\left(K_{1a}^+(\varepsilon_j)\ell_{aj}(0)\ell_{ak}(\varepsilon_j-\varepsilon_k)K_{1a}^-(\varepsilon_j)\right).$$

Thus, we have

$$\begin{aligned}
\tau_j &= \sum_{k\neq j}^{\mathcal{L}}\frac{\mathrm{tr}_a\left(K_{1a}^+(\varepsilon_j)\ell_{ak}(\varepsilon_j-\varepsilon_k)\ell_{aj}(0)K_{1a}^-(\varepsilon_j)\right)}{\sinh(\varepsilon_j-\varepsilon_k)} + \\
&\quad + \sum_{k=1}^{\mathcal{L}}\frac{\mathrm{tr}_a\left(K_{1a}^+(\varepsilon_j)\ell_{aj}(0)K_{1a}^-(\varepsilon_j)\ell_{ak}(\varepsilon_j+\varepsilon_k)\right)}{\sinh(\varepsilon_j+\varepsilon_k)} + \\
&\quad + \mathrm{tr}_a\left(K_{2a}^+(\varepsilon_j)\ell_{aj}(0)K_{1a}^-(\varepsilon_j)\right) + \mathrm{tr}_a\left(K_{1a}^+(\varepsilon_j)\ell_{aj}(0)K_{2a}^-(\varepsilon_j)\right).
\end{aligned}$$

Finally, after computing traces, we obtain

$$\tau_j = \sum_{k\neq j}^{\mathcal{L}}\frac{\sinh(\varepsilon_j+\xi)\sinh(\varepsilon_j-\xi)}{\sinh(\varepsilon_j-\varepsilon_k)}\left(2\cosh(\varepsilon_j-\varepsilon_k)S_k^zS_j^z + S_k^-S_j^+ + S_k^+S_j^-\right) +$$

$$+\sum_{k=1}^{\mathcal{L}} \frac{1}{\sinh(\varepsilon_j+\varepsilon_k)} \Big(2\sinh(\varepsilon_j+\xi)\sinh(\varepsilon_j-\xi)\cosh(\varepsilon_j+\varepsilon_k)S_j^z S_k^z - \\ - \sinh^2(\varepsilon_j+\xi)S_j^- S_k^+ - \sinh^2(\varepsilon_j-\xi)S_j^+ S_k^-\Big) + \\ + \left(\alpha\sinh(2\varepsilon_j) - \frac{1}{2}\sinh(2\xi)\right) S_j^z + \left(\beta\sinh(2\varepsilon_j) - \frac{1}{2}\sinh(2\xi)\right) S_j^z.$$

We rescale and denote $\tau_j^{trig} = \dfrac{\tau_j}{\sinh(\varepsilon_j+\xi)\sinh(\varepsilon_j-\xi)}$, so that

$$\tau_j^{trig} = \sum_{k\neq j}^{\mathcal{L}} \frac{1}{\sinh(\varepsilon_j-\varepsilon_k)} \left(2\cosh(\varepsilon_j-\varepsilon_k)S_j^z S_k^z + S_j^+ S_k^- + S_j^- S_k^+\right) + \\ + \sum_{k=1}^{\mathcal{L}} \frac{1}{\sinh(\varepsilon_j+\varepsilon_k)} \left(2\cosh(\varepsilon_j+\varepsilon_k)S_j^z S_k^z - \frac{\sinh(\varepsilon_j-\xi)}{\sinh(\varepsilon_j+\xi)} S_j^+ S_k^- - \\ - \frac{\sinh(\varepsilon_j+\xi)}{\sinh(\varepsilon_j-\xi)} S_j^- S_k^+\right) + \\ + \frac{(\alpha+\beta)\sinh(2\varepsilon_j) - \sinh(2\xi)}{\sinh(\varepsilon_j+\xi)\sinh(\varepsilon_j-\xi)} S_j^z. \tag{3.16}$$

Thus, $\{\tau_j^{trig},\ j = 1,\dots,\mathcal{L}\}$ are the mutually commuting conserved operators for **Trig. BQISM**.

3.2.2 The second family of conserved operators

Note that we have only considered one of two families of conserved operators. The second family is constructed as follows:

$$\lim_{u\to-\varepsilon_j} (u+\varepsilon_j)t(u) = \eta^2\tilde{\tau}_j + \mathcal{O}(\eta^3). \tag{3.17}$$

Proposition 3.2. *We have $\tilde{\tau}_j = -\tau_j$, where τ_j are given by (3.14) and $\tilde{\tau}_j$ by (3.17). Thus, the two families of conserved operators are equivalent.*

Proof. It is convenient to include the dependence on inhomogeneity parameters ε_j explicitly into the notation:

$$t(u,\vec{\varepsilon}) = \mathrm{tr}_a\left(K_a^+(u)L_{a\mathcal{L}}(u-\varepsilon_{\mathcal{L}})\cdots L_{a1}(u-\varepsilon_1)K_a^-(u)L_{a1}(u+\varepsilon_1)\cdots L_{a\mathcal{L}}(u+\varepsilon_{\mathcal{L}})\right).$$

Consider $t(u,\vec{\varepsilon})^T$, where $T = t_1 \cdots t_{\mathcal{L}}$ denotes the transpose over all spaces. Using

$$(\mathrm{tr}_a\, A_a)^{t_1\cdots t_{\mathcal{L}}} = \mathrm{tr}_a\left(A_a^{t_1\cdots t_{\mathcal{L}}}\right) = \mathrm{tr}_a\left(A_a^{t_a t_1\cdots t_{\mathcal{L}}}\right)$$

and the fact that all operators are symmetric, we have

$$\begin{aligned} t(u,\vec{\varepsilon})^T &= \mathrm{tr}_a\Big(L_{a\mathcal{L}}(u+\varepsilon_{\mathcal{L}})\cdots L_{a1}(u+\varepsilon_1)K_a^-(u)L_{a1}(u-\varepsilon_1)\cdots L_{a\mathcal{L}}(u-\varepsilon_{\mathcal{L}})K_a^+(u)\Big) = \\ &= \mathrm{tr}_a\Big(K_a^+(u)L_{a\mathcal{L}}(u+\varepsilon_{\mathcal{L}})\cdots L_{a1}(u+\varepsilon_1)K_a^-(u)L_{a1}(u-\varepsilon_1)\cdots L_{a\mathcal{L}}(u-\varepsilon_{\mathcal{L}})\Big) = \\ &= t(u,-\vec{\varepsilon}). \end{aligned}$$

It follows that

$$\lim_{u\to-\varepsilon_j}(u+\varepsilon_j)t(u,\vec{\varepsilon}) = \lim_{u\to-\varepsilon_j}(u+\varepsilon_j)t(u,-\vec{\varepsilon})^T.$$

Thus,

$$\tilde{\tau}_j(\vec{\varepsilon}) = \tau_j(-\vec{\varepsilon})^T = \sinh(\varepsilon_j+\xi)\sinh(\varepsilon_j-\xi)\tau_j^{trig}(-\vec{\varepsilon})^T.$$

Let's calculate $\tau_j^{trig}(-\vec{\varepsilon})^T$ from (3.16):

$$\begin{aligned} \tau_j^{trig}(-\vec{\varepsilon})^T = &-\sum_{k\neq j}^{\mathcal{L}}\frac{1}{\sinh(\varepsilon_j-\varepsilon_k)}\left(2\cosh(\varepsilon_j-\varepsilon_k)S_j^zS_k^z + S_j^-S_k^+ + S_j^+S_k^-\right) - \\ &-\sum_{k=1}^{\mathcal{L}}\frac{1}{\sinh(\varepsilon_j+\varepsilon_k)}\Bigg(2\cosh(\varepsilon_j+\varepsilon_k)S_j^zS_k^z - \frac{\sinh(\varepsilon_j+\xi)}{\sinh(\varepsilon_j-\xi)}S_k^+S_j^- - \\ &\qquad - \frac{\sinh(\varepsilon_j-\xi)}{\sinh(\varepsilon_j+\xi)}S_k^-S_j^+\Bigg) - \\ &-\frac{(\alpha+\beta)\sinh(2\varepsilon_j)+\sinh(2\xi)}{\sinh(\varepsilon_j-\xi)\sinh(\varepsilon_j+\xi)}S_j^z = \\ = &-\sum_{k\neq j}^{\mathcal{L}}\frac{1}{\sinh(\varepsilon_j-\varepsilon_k)}\left(2\cosh(\varepsilon_j-\varepsilon_k)S_j^zS_k^z + S_j^+S_k^- + S_j^-S_k^+\right) - \\ &-\sum_{k\neq j}^{\mathcal{L}}\frac{1}{\sinh(\varepsilon_j+\varepsilon_k)}\Bigg(2\cosh(\varepsilon_j+\varepsilon_k)S_j^zS_k^z - \frac{\sinh(\varepsilon_j-\xi)}{\sinh(\varepsilon_j+\xi)}S_j^+S_k^- - \\ &\qquad - \frac{\sinh(\varepsilon_j+\xi)}{\sinh(\varepsilon_j-\xi)}S_j^-S_k^+\Bigg) - \\ &-\frac{2\cosh(2\varepsilon_j)}{\sinh(2\varepsilon_j)}(S_j^z)^2 + \frac{1}{\sinh(2\varepsilon_j)}\left(\frac{\sinh(\varepsilon_j+\xi)}{\sinh(\varepsilon_j-\xi)}S_j^+S_j^- + \frac{\sinh(\varepsilon_j-\xi)}{\sinh(\varepsilon_j+\xi)}S_j^-S_j^+\right) - \\ &-\frac{(\alpha+\beta)\sinh(2\varepsilon_j)}{\sinh(\varepsilon_j-\xi)\sinh(\varepsilon_j+\xi)}S_j^z - \frac{\sinh(2\xi)}{\sinh(\varepsilon_j-\xi)\sinh(\varepsilon_j+\xi)}S_j^z. \end{aligned}$$

Consider

$$\begin{aligned}\tau_j^{trig}(\vec{\varepsilon}) + \tau_j^{trig}(-\vec{\varepsilon})^T &= \frac{1}{\sinh(2\varepsilon_j)}\left(\frac{\sinh(\varepsilon_j+\xi)}{\sinh(\varepsilon_j-\xi)} - \frac{\sinh(\varepsilon_j-\xi)}{\sinh(\varepsilon_j+\xi)}\right)[S_j^+, S_j^-] - \\ &- \frac{2\sinh(2\xi)}{\sinh(\varepsilon_j-\xi)\sinh(\varepsilon_j+\xi)}S_j^z = \\ &= \frac{2S_j^z}{\sinh(2\varepsilon_j)}\frac{\sinh^2(\varepsilon_j+\xi)-\sinh^2(\varepsilon_j-\xi)}{\sinh(\varepsilon_j-\xi)\sinh(\varepsilon_j+\xi)} - \\ &- \frac{2\sinh(2\xi)S_j^z}{\sinh(\varepsilon_j-\xi)\sinh(\varepsilon_j+\xi)} = 0.\end{aligned}$$

From this it follows that $\tilde{\tau}_j(\vec{\varepsilon}) = -\tau_j(\vec{\varepsilon})$. □

Thus, we have shown that the second family of conserved operators obtained from (3.17) is equivalent to the first one from (3.14).

3.2.3 Variable change #1

The variable change #1 (3.5), particularly $\varepsilon_j \mapsto \varepsilon_j + \rho/2$, gives the conserved operators for **Trig. BQISM′**:

$$\begin{aligned}\tau_j^{trig'} &= \sum_{k\neq j}^{\mathcal{L}} \frac{1}{\sinh(\varepsilon_j-\varepsilon_k)}\left(2\cosh(\varepsilon_j-\varepsilon_k)S_j^zS_k^z + S_j^+S_k^- + S_j^-S_k^+\right) + \\ &+ \sum_{k=1}^{\mathcal{L}} \frac{1}{\sinh(\varepsilon_j+\varepsilon_k+\rho)}\Bigg(2\cosh(\varepsilon_j+\varepsilon_k+\rho)S_j^zS_k^z - \\ &\quad - \frac{\sinh(\varepsilon_j+\rho/2-\xi)}{\sinh(\varepsilon_j+\rho/2+\xi)}S_j^+S_k^- - \frac{\sinh(\varepsilon_j+\rho/2+\xi)}{\sinh(\varepsilon_j+\rho/2-\xi)}S_j^-S_k^+\Bigg) + \\ &+ \frac{(\alpha+\beta)\sinh(2\varepsilon_j+\rho) - \sinh(2\xi)}{\sinh(\varepsilon_j+\rho/2+\xi)\sinh(\varepsilon_j+\rho/2-\xi)}S_j^z.\end{aligned} \tag{3.18}$$

3.2.4 Attenuated limit

Taking $\rho \to \infty$ in (3.18) yields the conserved operators for **Trig. QISM** (2.26):

$$\tau_j^{trig'} \to \sum_{k\neq j}^{\mathcal{L}} \frac{1}{\sinh(\varepsilon_j-\varepsilon_k)}\left(2\cosh(\varepsilon_j-\varepsilon_k)S_j^zS_k^z + S_j^+S_k^- + S_j^-S_k^+\right) - 2\gamma S_j^z,$$

where $\gamma = -(\alpha+\beta+N-\mathcal{L}/2)$.

3.2.5 Rational limit

The rational limit of the conserved operators for **Trig. BQISM** (3.16) gives the conserved operators for **Rat. BQISM**:

$$
\begin{aligned}
\tau_j^{rat} = & \sum_{k\neq j}^{\mathcal{L}} \frac{1}{\varepsilon_j - \varepsilon_k} \left(2S_j^z S_k^z + S_j^+ S_k^- + S_j^- S_k^+\right) + \\
& + \sum_{k=1}^{\mathcal{L}} \frac{1}{\varepsilon_j + \varepsilon_k} \left(2S_j^z S_k^z - \frac{\varepsilon_j - \xi}{\varepsilon_j + \xi} S_j^+ S_k^- - \frac{\varepsilon_j + \xi}{\varepsilon_j - \xi} S_j^- S_k^+\right) + \\
& + \frac{2(\alpha+\beta)\varepsilon_j - 2\xi}{\varepsilon_j^2 - \xi^2} S_j^z.
\end{aligned} \tag{3.19}
$$

We rewrite this expression as

$$
\begin{aligned}
\tau_j^{rat} = & \sum_{k\neq j}^{\mathcal{L}} \left(\frac{1}{\varepsilon_j - \varepsilon_k} + \frac{1}{\varepsilon_j + \varepsilon_k}\right) 2S_j^z S_k^z + \sum_{k\neq j}^{\mathcal{L}} \left(\frac{1}{\varepsilon_j - \varepsilon_k} - \frac{1}{\varepsilon_j + \varepsilon_k}\frac{\varepsilon_j - \xi}{\varepsilon_j + \xi}\right) S_j^+ S_k^- + \\
& + \sum_{k\neq j}^{\mathcal{L}} \left(\frac{1}{\varepsilon_j - \varepsilon_k} - \frac{1}{\varepsilon_j + \varepsilon_k}\frac{\varepsilon_j + \xi}{\varepsilon_j - \xi}\right) S_j^- S_k^+ + \frac{1}{2\varepsilon_j} 2(S_j^z)^2 - \\
& - \frac{1}{2\varepsilon_j}\frac{\varepsilon_j - \xi}{\varepsilon_j + \xi} S_j^+ S_j^- - \frac{1}{2\varepsilon_j}\frac{\varepsilon_j + \xi}{\varepsilon_j - \xi} S_j^- S_j^+ + \frac{2(\alpha+\beta)\varepsilon_j}{\varepsilon_j^2 - \xi^2} S_j^z - \frac{2\xi}{\varepsilon_j^2 - \xi^2} S_j^z.
\end{aligned}
$$

In the spin-1/2 representation (2.16) we can use the following identities:

$$
S^+ S^- = \frac{I}{2} + S^z, \;\; S^- S^+ = \frac{I}{2} - S^z, \;\; (S^z)^2 = \frac{I}{4}.
$$

Also note the following:

$$
\begin{aligned}
\frac{1}{\varepsilon_j - \varepsilon_k} - \frac{1}{\varepsilon_j + \varepsilon_k}\frac{\varepsilon_j - \xi}{\varepsilon_j + \xi} &= \frac{2\varepsilon_j(\varepsilon_k + \xi)}{(\varepsilon_j^2 - \varepsilon_k^2)(\varepsilon_j + \xi)}, \\
\frac{1}{\varepsilon_j - \varepsilon_k} - \frac{1}{\varepsilon_j + \varepsilon_k}\frac{\varepsilon_j + \xi}{\varepsilon_j - \xi} &= \frac{2\varepsilon_j(\varepsilon_k - \xi)}{(\varepsilon_j^2 - \varepsilon_k^2)(\varepsilon_j - \xi)}.
\end{aligned}
$$

Then we obtain

$$
\begin{aligned}
\tau_j^{rat} = & \sum_{k\neq j}^{\mathcal{L}} \frac{4\varepsilon_j}{\varepsilon_j^2 - \varepsilon_k^2} S_j^z S_k^z + \sum_{k\neq j}^{\mathcal{L}} \frac{2\varepsilon_j}{\varepsilon_j^2 - \varepsilon_k^2} \left(\frac{\varepsilon_k + \xi}{\varepsilon_j + \xi} S_j^+ S_k^- + \frac{\varepsilon_k - \xi}{\varepsilon_j - \xi} S_j^- S_k^+\right) + \\
& + \frac{I}{4\varepsilon_j} - \frac{1}{2\varepsilon_j}\frac{\varepsilon_j - \xi}{\varepsilon_j + \xi}\left(\frac{I}{2} + S^z\right) - \frac{1}{2\varepsilon_j}\frac{\varepsilon_j + \xi}{\varepsilon_j - \xi}\left(\frac{I}{2} - S^z\right) +
\end{aligned}
$$

$$+\frac{2(\alpha+\beta)\varepsilon_j}{\varepsilon_j^2-\xi^2}S_j^z-\frac{2\xi}{\varepsilon_j^2-\xi^2}S_j^z=$$
$$=\sum_{k\neq j}^{\mathcal{L}}\frac{4\varepsilon_j}{\varepsilon_j^2-\varepsilon_k^2}S_j^zS_k^z+\sum_{k\neq j}^{\mathcal{L}}\frac{2\varepsilon_j}{\varepsilon_j^2-\varepsilon_k^2}\left(\frac{\varepsilon_k+\xi}{\varepsilon_j+\xi}S_j^+S_k^-+\frac{\varepsilon_k-\xi}{\varepsilon_j-\xi}S_j^-S_k^+\right)+$$
$$+\frac{I}{4\varepsilon_j}-\frac{\varepsilon_j^2+\xi^2}{\varepsilon_j^2-\xi^2}\frac{I}{2\varepsilon_j}+\frac{2(\alpha+\beta)\varepsilon_j}{\varepsilon_j^2-\xi^2}S_j^z.$$

Multiplying by ε_j we obtain

$$\begin{aligned}\varepsilon_j\tau_j^{rat}=&\sum_{k\neq j}^{\mathcal{L}}\frac{4\varepsilon_j^2}{\varepsilon_j^2-\varepsilon_k^2}S_j^zS_k^z+\sum_{k\neq j}^{\mathcal{L}}\frac{2\varepsilon_j^2}{\varepsilon_j^2-\varepsilon_k^2}\left(\frac{\varepsilon_k+\xi}{\varepsilon_j+\xi}S_j^+S_k^-+\frac{\varepsilon_k-\xi}{\varepsilon_j-\xi}S_j^-S_k^+\right)+\\&+\frac{I}{4}-\frac{\varepsilon_j^2+\xi^2}{\varepsilon_j^2-\xi^2}\frac{I}{2}+\frac{2(\alpha+\beta)\varepsilon_j^2}{\varepsilon_j^2-\xi^2}S_j^z.\end{aligned} \tag{3.20}$$

3.2.6 Rational BQISM and trigonometric QISM equivalence

Substitute

$$\begin{aligned}\sum_{k\neq j}^{\mathcal{L}}\frac{4\varepsilon_j^2}{\varepsilon_j^2-\varepsilon_k^2}S_j^zS_k^z&=2\sum_{k\neq j}^{\mathcal{L}}\left(\frac{\varepsilon_j^2+\varepsilon_k^2}{\varepsilon_j^2-\varepsilon_k^2}+1\right)S_j^zS_k^z=\\&=2\sum_{k\neq j}^{\mathcal{L}}\frac{\varepsilon_j^2+\varepsilon_k^2}{\varepsilon_j^2-\varepsilon_k^2}S_j^zS_k^z+2\left(N-\frac{\mathcal{L}}{2}\right)S_j^z-\frac{I}{2}\end{aligned}$$

into (3.20) to obtain

$$\begin{aligned}\varepsilon_j\tau_j^{rat}=&2\sum_{k\neq j}^{\mathcal{L}}\frac{\varepsilon_j^2+\varepsilon_k^2}{\varepsilon_j^2-\varepsilon_k^2}S_j^zS_k^z+\sum_{k\neq j}^{\mathcal{L}}\frac{2\varepsilon_j^2}{\varepsilon_j^2-\varepsilon_k^2}\left(\frac{\varepsilon_k+\xi}{\varepsilon_j+\xi}S_j^+S_k^-+\frac{\varepsilon_k-\xi}{\varepsilon_j-\xi}S_j^-S_k^+\right)+\\&+2\left(N-\frac{\mathcal{L}}{2}+\frac{\varepsilon_j^2}{\varepsilon_j^2-\xi^2}(\alpha+\beta)\right)S_j^z-\left(\frac{1}{4}+\frac{1}{2}\frac{\varepsilon_j^2+\xi^2}{\varepsilon_j^2-\xi^2}\right)I.\end{aligned}$$

Setting $\xi=0$ we obtain

$$\begin{aligned}\varepsilon_j\tau_j^{rat}\big|_{\xi=0}=&2\sum_{k\neq j}^{\mathcal{L}}\frac{\varepsilon_j^2+\varepsilon_k^2}{\varepsilon_j^2-\varepsilon_k^2}S_j^zS_k^z+\sum_{k\neq j}^{\mathcal{L}}\frac{2\varepsilon_j\varepsilon_k}{\varepsilon_j^2-\varepsilon_k^2}\left(S_j^+S_k^-+S_j^-S_k^+\right)+\\&+2\left(\alpha+\beta+N-\frac{\mathcal{L}}{2}\right)S_j^z-\frac{3I}{4}.\end{aligned}$$

The variable change $\varepsilon_j \mapsto \exp \varepsilon_j$ gives **Trig. QISM** (2.26) with $\gamma = -(\alpha+\beta+N-\mathcal{L}/2)$ (up to a constant term $-3I/4$):

$$\varepsilon_j \tau_j^{rat}\big|_{\xi=0} \mapsto 2\sum_{k\neq j}^{\mathcal{L}} \coth(\varepsilon_j - \varepsilon_k) S_j^z S_k^z + \sum_{k\neq j}^{\mathcal{L}} \frac{1}{\sinh(\varepsilon_j-\varepsilon_k)} \left(S_j^+ S_k^- + S_j^- S_k^+\right) + \\ + 2\left(\alpha+\beta+N-\frac{\mathcal{L}}{2}\right) S_j^z - \frac{3I}{4}.$$

Now let us start with **Trig. QISM** (2.26) and show that it can be mapped back into (3.20). First of all, let us make a change of variables $\varepsilon_j = \ln z_j$:

$$\tau_j^{(1)} = 2\sum_{k\neq j}^{\mathcal{L}} \frac{z_j^2+z_k^2}{z_j^2-z_k^2} S_j^z S_k^z + \sum_{k\neq j}^{\mathcal{L}} \frac{2z_j z_k}{z_j^2-z_k^2}\left(S_j^+ S_k^- + S_j^- S_k^+\right) - 2\gamma S_j^z.$$

Using $\dfrac{z_j^2+z_k^2}{z_j^2-z_k^2} = \dfrac{2z_j^2}{z_j^2-z_k^2} - 1$ we obtain

$$\tau_j^{(1)} = -2\sum_{k\neq j}^{\mathcal{L}} S_j^z S_k^z + 4\sum_{k\neq j}^{\mathcal{L}} \frac{z_j^2}{z_j^2-z_k^2} S_j^z S_k^z + 2\sum_{k\neq j}^{\mathcal{L}} \frac{z_j z_k}{z_j^2-z_k^2}\left(S_j^+ S_k^- + S_j^- S_k^+\right) - 2\gamma S_j^z.$$

Furthermore, since

$$2\sum_{k\neq j}^{\mathcal{L}} S_j^z S_k^z = 2\left(N - \frac{\mathcal{L}}{2} - S_j^z\right) S_j^z = 2\left(N-\frac{\mathcal{L}}{2}\right) S_j^z - 2(S_j^z)^2$$

and $(S^z)^2 = I/4$ for the spin-1/2 representation, we obtain

$$\tau_j^{(1)} = 4\sum_{k\neq j}^{\mathcal{L}} \frac{z_j^2}{z_j^2-z_k^2} S_j^z S_k^z + 2\sum_{k\neq j}^{\mathcal{L}} \frac{z_k z_j}{z_j^2-z_k^2}\left(S_j^+ S_k^- + S_j^- S_k^+\right) - 2\left(\gamma + N - \frac{\mathcal{L}}{2}\right) S_j^z + \frac{I}{2}.$$

A change of variable $z_j \mapsto \sqrt{\varepsilon_j^2 - \xi^2}$ gives the following conserved operators:

$$\tau_j^{(2)} = 4\sum_{k\neq j}^{\mathcal{L}} \frac{\varepsilon_j^2-\xi^2}{\varepsilon_j^2-\varepsilon_k^2} S_j^z S_k^z + 2\sum_{k\neq j}^{\mathcal{L}} \frac{\sqrt{\varepsilon_j^2-\xi^2}\sqrt{\varepsilon_k^2-\xi^2}}{\varepsilon_j^2-\varepsilon_k^2}\left(S_j^+ S_k^- + S_j^- S_k^+\right) - \\ - 2\left(\gamma + N - \frac{\mathcal{L}}{2}\right) S_j^z + \frac{I}{2}.$$

Note that up to this point, all we have done is apply the change of variables given in (3.10) on the ε_j. We further rescale each conserved operator $\tau_j^{(2)}$ by the factor $\dfrac{\varepsilon_j^2}{\varepsilon_j^2-\xi^2}$:

$$\tau_j^{(3)} = 4\sum_{k\neq j}^{\mathcal{L}} \frac{\varepsilon_j^2}{\varepsilon_j^2-\varepsilon_k^2} S_j^z S_k^z + 2\sum_{k\neq j}^{\mathcal{L}} \frac{\varepsilon_j^2\sqrt{\varepsilon_k^2-\xi^2}}{(\varepsilon_j^2-\varepsilon_k^2)\sqrt{\varepsilon_j^2-\xi^2}} \left(S_j^+S_k^- + S_j^-S_k^+\right) - \\ -2\left(\gamma+N-\frac{\mathcal{L}}{2}\right)\frac{\varepsilon_j^2}{\varepsilon_j^2-\xi^2}S_j^z + \frac{\varepsilon_j^2}{\varepsilon_j^2-\xi^2}\frac{I}{2}.$$

Consider a local transformation on the jth space in the tensor product

$$U_j = \operatorname{diag}\left(\sqrt{\frac{\varepsilon_j-\xi}{\varepsilon_j+\xi}}, 1\right).$$

Under these transformations we have

$$U_j S_j^z U_j^{-1} = S_j^z,$$
$$U_j S_j^+ U_j^{-1} = \sqrt{\frac{\varepsilon_j-\xi}{\varepsilon_j+\xi}} S_j^+,$$
$$U_j S_j^- U_j^{-1} = \sqrt{\frac{\varepsilon_j+\xi}{\varepsilon_j-\xi}} S_j^-.$$

Under the global transformation $U = U_1U_2\cdots U_{\mathcal{L}}$ we find

$$\tau_j^{(4)} = U\tau_j^{(3)}U^{-1} = \\ = 4\sum_{k\neq j}^{\mathcal{L}} \frac{\varepsilon_j^2}{\varepsilon_j^2-\varepsilon_k^2} S_j^z S_k^z + 2\sum_{k\neq j}^{\mathcal{L}} \frac{\varepsilon_j^2}{\varepsilon_j^2-\varepsilon_k^2}\left(\frac{\varepsilon_k+\xi}{\varepsilon_j+\xi}S_j^+S_k^- + \frac{\varepsilon_k-\xi}{\varepsilon_j-\xi}S_j^-S_k^+\right) - \\ -2\left(\gamma+N-\frac{\mathcal{L}}{2}\right)\frac{\varepsilon_j^2}{\varepsilon_j^2-\xi^2}S_j^z + \frac{\varepsilon_j^2}{\varepsilon_j^2-\xi^2}\frac{I}{2}.$$

These are the same as $\varepsilon_j\tau_j^{rat}$ **Rat. BQISM** (3.20), up to the constant term, taking into account that $\gamma = -(\alpha+\beta+N-\mathcal{L}/2)$. Thus, we have

$$\tau_j^{(4)} - \varepsilon_j\tau_j^{rat} = \frac{\varepsilon_j^2}{\varepsilon_j^2-\xi^2}\frac{I}{2} + \frac{\varepsilon_j^2+\xi^2}{\varepsilon_j^2-\xi^2}\frac{I}{2} - \frac{I}{4}.$$

Finally, we can obtain

$$\tau_j^{rat} = \frac{1}{\varepsilon_j}\left(\tau_j^{(4)} - \frac{\varepsilon_j^2}{\varepsilon_j^2-\xi^2}\frac{I}{2} - \frac{\varepsilon_j^2+\xi^2}{\varepsilon_j^2-\xi^2}\frac{I}{2} + \frac{I}{4}\right).$$

3.2.7 Variable change #2, rescaling, and a basis transformation

Our goal now is to demonstrate how to transform **Rat. BQISM** (3.19) back into **Trig. BQISM** (3.16). First of all, we make a change of variables $\varepsilon_j = \ln z_j,\ \xi = \ln\chi$ in (3.16):

$$\begin{aligned}
\tau_j^{trig} &= \sum_{k\neq j}^{\mathcal{L}}\left(2\frac{z_j^2+z_k^2}{z_j^2-z_k^2}S_j^zS_k^z + \frac{2z_jz_k}{z_j^2-z_k^2}\left(S_j^+S_k^- + S_j^-S_k^+\right)\right) + \\
&+ \sum_{k=1}^{\mathcal{L}}\left(2\frac{z_j^2z_k^2+1}{z_j^2z_k^2-1}S_j^zS_k^z - \frac{2z_jz_k}{z_j^2z_k^2-1}\left(\frac{z_j^2-\chi^2}{\chi^2z_j^2-1}S_j^+S_k^- + \frac{\chi^2z_j^2-1}{z_j^2-\chi^2}S_j^-S_k^+\right)\right) + \\
&+ 2\frac{(\alpha+\beta)\chi^2(z_j^4-1) - z_j^2(\chi^4-1)}{(\chi^2z_j^2-1)(z_j^2-\chi^2)}S_j^z = \\
&= 2\sum_{k\neq j}^{\mathcal{L}}\left(\frac{z_j^2+z_k^2}{z_j^2-z_k^2} + \frac{z_j^2z_k^2+1}{z_j^2z_k^2-1}\right)S_j^zS_k^z + \\
&+ 2\sum_{k\neq j}^{\mathcal{L}}\left[\left(\frac{z_jz_k}{z_j^2-z_k^2} - \frac{z_jz_k}{z_j^2z_k^2-1}\frac{z_j^2-\chi^2}{\chi^2z_j^2-1}\right)S_j^+S_k^- + \right. \\
&\qquad\left. + \left(\frac{z_jz_k}{z_j^2-z_k^2} - \frac{z_jz_k}{z_j^2z_k^2-1}\frac{\chi^2z_j^2-1}{z_j^2-\chi^2}\right)S_j^-S_k^+\right] + \\
&+ \frac{z_j^4+1}{z_j^4-1}\frac{I}{2} - \frac{2z_j^2}{z_j^4-1}\frac{z_j^2-\chi^2}{\chi^2z_j^2-1}S_j^+S_j^- - \frac{2z_j^2}{z_j^4-1}\frac{\chi^2z_j^2-1}{z_j^2-\chi^2}S_j^-S_j^+ + \\
&+ \frac{2(\alpha+\beta)\chi^2(z_j^4-1)}{(\chi^2z_j^2-1)(z_j^2-\chi^2)}S_j^z - \frac{2z_j^2(\chi^4-1)}{(\chi^2z_j^2-1)(z_j^2-\chi^2)}S_j^z.
\end{aligned}$$

Using $S^+S^- = I/2 + S^z,\ S^-S^+ = I/2 - S^z$ and simplifying we obtain

$$\begin{aligned}
\tau_j^{trig} &= 2\sum_{k\neq j}^{\mathcal{L}}\left(\frac{z_j^2+z_k^2}{z_j^2-z_k^2} + \frac{z_j^2z_k^2+1}{z_j^2z_k^2-1}\right)S_j^zS_k^z + \\
&+ 2\sum_{k\neq j}^{\mathcal{L}}\frac{z_jz_k(z_j^4-1)}{(z_j^2-z_k^2)(z_j^2z_k^2-1)}\left[\frac{z_k^2\chi^2-1}{z_j^2\chi^2-1}S_j^+S_k^- + \frac{z_k^2-\chi^2}{z_j^2-\chi^2}S_j^-S_k^+\right] + \\
&+ \frac{z_j^4+1}{z_j^4-1}\frac{I}{2} - \frac{z_j^2}{z_j^4-1}\left(\frac{z_j^2-\chi^2}{\chi^2z_j^2-1} + \frac{\chi^2z_j^2-1}{z_j^2-\chi^2}\right)I + \frac{2(\alpha+\beta)\chi^2(z_j^4-1)}{(\chi^2z_j^2-1)(z_j^2-\chi^2)}S_j^z.
\end{aligned} \tag{3.21}$$

We begin with the the form (3.20) of **Rat. BQISM**, multiplied by ε_j:

$$\tilde{\tau}_j^{(1)} = \varepsilon_j \tau_j^{rat} = \sum_{k\neq j}^{\mathcal{L}} \frac{4\varepsilon_j^2}{\varepsilon_j^2 - \varepsilon_k^2} S_j^z S_k^z + \sum_{k\neq j}^{\mathcal{L}} \frac{2\varepsilon_j^2}{\varepsilon_j^2 - \varepsilon_k^2} \left(\frac{\varepsilon_k + \xi}{\varepsilon_j + \xi} S_j^+ S_k^- + \frac{\varepsilon_k - \xi}{\varepsilon_j - \xi} S_j^- S_k^+ \right) + \\ + \frac{I}{4} - \frac{\varepsilon_j^2 + \xi^2}{\varepsilon_j^2 - \xi^2} \frac{I}{2} + \frac{2(\alpha+\beta)\varepsilon_j^2}{\varepsilon_j^2 - \xi^2} S_j^z.$$

Now, let us make a change of variables $\varepsilon_j \mapsto \dfrac{z_j - z_j^{-1}}{2}$, $\xi \mapsto \dfrac{\chi - \chi^{-1}}{2}$:

$$\tilde{\tau}_j^{(2)} = \sum_{k\neq j}^{\mathcal{L}} \frac{4(z_j - z_j^{-1})^2}{(z_j - z_j^{-1})^2 - (z_k - z_k^{-1})^2} S_j^z S_k^z + \\ + \sum_{k\neq j}^{\mathcal{L}} \frac{2(z_j - z_j^{-1})^2}{(z_j - z_j^{-1})^2 - (z_k - z_k^{-1})^2} \times \\ \times \left(\frac{z_k - z_k^{-1} + \chi - \chi^{-1}}{z_j - z_j^{-1} + \chi - \chi^{-1}} S_j^+ S_k^- + \frac{z_k - z_k^{-1} - \chi + \chi^{-1}}{z_j - z_j^{-1} - \chi + \chi^{-1}} S_j^- S_k^+ \right) + \\ + \frac{I}{4} - \frac{(z_j - z_j^{-1})^2 + (\chi - \chi^{-1})^2}{(z_j - z_j^{-1})^2 - (\chi - \chi^{-1})^2} \frac{I}{2} + \frac{2(\alpha+\beta)(z_j - z_j^{-1})^2}{(z_j - z_j^{-1})^2 - (\chi - \chi^{-1})^2} S_j^z.$$

Then, rescale by $\dfrac{z_j^2 - z_j^{-2}}{(z_j - z_j^{-1})^2}$:

$$\tilde{\tau}_j^{(3)} = \sum_{k\neq j}^{\mathcal{L}} \frac{4(z_j^2 - z_j^{-2})}{(z_j - z_j^{-1})^2 - (z_k - z_k^{-1})^2} S_j^z S_k^z + \\ + \sum_{k\neq j}^{\mathcal{L}} \frac{2(z_j^2 - z_j^{-2})}{(z_j - z_j^{-1})^2 - (z_k - z_k^{-1})^2} \times \\ \times \left(\frac{z_k - z_k^{-1} + \chi - \chi^{-1}}{z_j - z_j^{-1} + \chi - \chi^{-1}} S_j^+ S_k^- + \frac{z_k - z_k^{-1} - \chi + \chi^{-1}}{z_j - z_j^{-1} - \chi + \chi^{-1}} S_j^- S_k^+ \right) + \\ + \frac{z_j^2 - z_j^{-2}}{(z_j - z_j^{-1})^2} \frac{I}{4} - \frac{(z_j - z_j^{-1})^2 + (\chi - \chi^{-1})^2}{(z_j - z_j^{-1})^2 - (\chi - \chi^{-1})^2} \frac{z_j^2 - z_j^{-2}}{(z_j - z_j^{-1})^2} \frac{I}{2} + \\ + \frac{2(\alpha+\beta)(z_j^2 - z_j^{-2})}{(z_j - z_j^{-1})^2 - (\chi - \chi^{-1})^2} S_j^z.$$

Using the identities

$$\frac{2(z_j^2 - z_j^{-2})}{(z_j - z_j^{-1})^2 - (z_k - z_k^{-1})^2} = \frac{z_j^2 + z_k^2}{z_j^2 - z_k^2} + \frac{z_j^2 z_k^2 + 1}{z_j^2 z_k^2 - 1} = \frac{2z_k^2(z_j^4 - 1)}{(z_j^2 - z_k^2)(z_j^2 z_k^2 - 1)},$$

$$\frac{2(z_j^2-\chi^{-2})}{(z_j-z_j^{-1})^2-(\chi-\chi^{-1})^2}=\frac{z_j^2+\chi^2}{z_j^2-\chi^2}+\frac{z_j^2\chi^2+1}{z_j^2\chi^2-1}=\frac{2\chi^2(z_j^4-1)}{(z_j^2-\chi^2)(z_j^2\chi^2-1)},$$

we obtain

$$\begin{aligned}\tilde{\tau}_j^{(3)}=&\,2\sum_{k\neq j}^{\mathcal{L}}\left(\frac{z_j^2+z_k^2}{z_j^2-z_k^2}+\frac{z_j^2z_k^2+1}{z_j^2z_k^2-1}\right)S_j^zS_k^z+\\&+2\sum_{k\neq j}^{\mathcal{L}}\frac{z_k^2(z_j^4-1)}{(z_j^2-z_k^2)(z_j^2z_k^2-1)}\times\\&\times\left(\frac{z_k-z_k^{-1}+\chi-\chi^{-1}}{z_j-z_j^{-1}+\chi-\chi^{-1}}S_j^+S_k^-+\frac{z_k-z_k^{-1}-\chi+\chi^{-1}}{z_j-z_j^{-1}-\chi+\chi^{-1}}S_j^-S_k^+\right)+\\&+\frac{z_j^2-z_j^{-2}}{(z_j-z_j^{-1})^2}\frac{I}{4}-\frac{(z_j-z_j^{-1})^2+(\chi-\chi^{-1})^2}{(z_j-z_j^{-1})^2}\frac{\chi^2(z_j^4-1)}{(z_j^2-\chi^2)(z_j^2\chi^2-1)}\frac{I}{2}+\\&+2(\alpha+\beta)\frac{\chi^2(z_j^4-1)}{(z_j^2-\chi^2)(z_j^2\chi^2-1)}S_j^z.\end{aligned}$$

Now we see that the first term already matches with the first term of (3.21). To match the second term we need to make a basis transformation of the type $U=U_1U_2\cdots U_{\mathcal{L}}$, where

$$U_j=\operatorname{diag}(x_j,1)$$

with

$$U_j^{-1}=\operatorname{diag}(x_j^{-1},1),$$

where $x_j\in\mathbb{C}$. Under these transformations we have

$$\begin{aligned}U_jS_j^zU_j^{-1}&=S_j^z,\\U_jS_j^+U_j^{-1}&=x_jS_j^+,\\U_jS_j^-U_j^{-1}&=x_j^{-1}S_j^-.\end{aligned}$$

To match the second sum in (3.21) we need x_j and x_k to satisfy

$$\begin{cases}\dfrac{z_jz_k(z_j^4-1)}{(z_j^2-z_k^2)(z_j^2z_k^2-1)}\dfrac{z_k^2\chi^2-1}{z_j^2\chi^2-1}=\dfrac{z_k^2(z_j^4-1)}{(z_j^2-z_k^2)(z_j^2z_k^2-1)}\dfrac{z_k-z_k^{-1}+\chi-\chi^{-1}}{z_j-z_j^{-1}+\chi-\chi^{-1}}x_jx_k^{-1},\\\dfrac{z_jz_k(z_j^4-1)}{(z_j^2-z_k^2)(z_j^2z_k^2-1)}\dfrac{z_k^2-\chi^2}{z_j^2-\chi^2}=\dfrac{z_k^2(z_j^4-1)}{(z_j^2-z_k^2)(z_j^2z_k^2-1)}\dfrac{z_k-z_k^{-1}-\chi+\chi^{-1}}{z_j-z_j^{-1}-\chi+\chi^{-1}}x_j^{-1}x_k,\end{cases}$$

or

$$\begin{cases} \dfrac{x_j}{x_k} = \dfrac{z_j(z_k^2\chi^2-1)(z_j-z_j^{-1}+\chi-\chi^{-1})}{z_k(z_j^2\chi^2-1)(z_k-z_k^{-1}+\chi-\chi^{-1})}, \\ \dfrac{x_j}{x_k} = \dfrac{z_k(z_j^2-\chi^2)(z_k-z_k^{-1}-\chi+\chi^{-1})}{z_j(z_k^2-\chi^2)(z_j-z_j^{-1}-\chi+\chi^{-1})}. \end{cases}$$

We may check that these equations are consistent, i.e.,

$$\frac{z_j(z_k^2\chi^2-1)(z_j-z_j^{-1}+\chi-\chi^{-1})}{z_k(z_j^2\chi^2-1)(z_k-z_k^{-1}+\chi-\chi^{-1})} = \frac{z_k(z_j^2-\chi^2)(z_k-z_k^{-1}-\chi+\chi^{-1})}{z_j(z_k^2-\chi^2)(z_j-z_j^{-1}-\chi+\chi^{-1})} \iff$$

$$\iff \frac{z_j^2}{z_k^2}\frac{(z_j-z_j^{-1})^2-(\chi-\chi^{-1})^2}{(z_k-z_k^{-1})^2-(\chi-\chi^{-1})^2} = \frac{(z_j^2-\chi^2)(z_j^2\chi^2-1)}{(z_k^2-\chi^2)(z_k^2\chi^2-1)} \iff$$

$$\iff 1=1.$$

We choose

$$x_j = \frac{z_j(z_j-z_j^{-1}+\chi-\chi^{-1})}{z_j^2\chi^2-1}.$$

Finally, we have

$$\begin{aligned}
\tilde{\tau}_j^{(4)} &= U\tilde{\tau}_j^{(3)}U^{-1} = \\
&= 2\sum_{k\neq j}^{\mathcal{L}}\left(\frac{z_j^2+z_k^2}{z_j^2-z_k^2}+\frac{z_j^2z_k^2+1}{z_j^2z_k^2-1}\right)S_j^zS_k^z + \\
&\quad + 2\sum_{k\neq j}^{\mathcal{L}}\frac{z_jz_k(z_j^4-1)}{(z_j^2-z_k^2)(z_j^2z_k^2-1)}\left[\frac{z_k^2\chi^2-1}{z_j^2\chi^2-1}S_j^+S_k^- + \frac{z_k^2-\chi^2}{z_j^2-\chi^2}S_j^-S_k^+\right] + \\
&\quad + \frac{z_j^2-z_j^{-2}}{(z_j-z_j^{-1})^2}\frac{I}{4} - \frac{(z_j-z_j^{-1})^2+(\chi-\chi^{-1})^2}{(z_j-z_j^{-1})^2}\frac{\chi^2(z_j^4-1)}{(z_j^2-\chi^2)(z_j^2\chi^2-1)}\frac{I}{2} + \\
&\quad + \frac{2(\alpha+\beta)\chi^2(z_j^4-1)}{(\chi^2z_j^2-1)(z_j^2-\chi^2)}S_j^z,
\end{aligned}$$

which is the same as τ_j^{trig} (3.21) up to the constant term:

$$\begin{aligned}
\tilde{\tau}_j^{(4)} - \tau_j^{trig} &= \frac{z_j^2-z_j^{-2}}{(z_j-z_j^{-1})^2}\frac{I}{4} - \frac{(z_j-z_j^{-1})^2+(\chi-\chi^{-1})^2}{(z_j-z_j^{-1})^2}\frac{\chi^2(z_j^4-1)}{(z_j^2-\chi^2)(z_j^2\chi^2-1)}\frac{I}{2} - \\
&\quad - \frac{z_j^4+1}{z_j^4-1}\frac{I}{2} + \frac{z_j^2}{z_j^4-1}\left(\frac{z_j^2-\chi^2}{\chi^2z_j^2-1}+\frac{\chi^2z_j^2-1}{z_j^2-\chi^2}\right)I.
\end{aligned}$$

3.2.8 Variable change #3, rescaling, and a basis transformation

As in the case of the BAE, variable change #3 is defined as the composition which leads to (3.13). Combined with the appropriate composition of basis transformations and rescalings described above, this leads to the following mappings for the conserved operators:

Trig. QISM (2.26) $\xrightarrow{3.2.6}$ **Rat. BQISM** (3.19) $\xrightarrow{3.2.7}$ **Trig. BQISM** (3.16) $\xrightarrow{3.2.3}$ **Trig. BQISM′** (3.18),

where the arrow labels refer to the subsections where the corresponding operations are described.

3.2.9 Reduction to the rational, twisted-periodic case

In the rational limit of **Trig. QISM** (2.26) we obtain the conserved operators **Rat. QISM** (2.30):

$$\tau_j = -2\gamma S_j^z + \sum_{k\neq j}^{\mathcal{L}} \frac{2S_j^z S_k^z + S_j^+ S_k^- + S_j^- S_k^+}{\varepsilon_j - \varepsilon_k}.$$

We can also obtain these conserved operators via the attenuated limit from **Rat. BQISM** (3.19). First introduce ρ by the variable change #1:

$$\begin{aligned}\tau_j^{rat'} = &\sum_{k\neq j}^{\mathcal{L}} \frac{1}{\varepsilon_j - \varepsilon_k}\left(2S_j^z S_k^z + S_j^+ S_k^- + S_j^- S_k^+\right) + \\ &+ \sum_{k=1}^{\mathcal{L}} \frac{1}{\varepsilon_j + \varepsilon_k + \rho}\left(2S_j^z S_k^z - \frac{\varepsilon_j + \rho/2 - \xi}{\varepsilon_j + \rho/2 + \xi} S_j^+ S_k^- - \frac{\varepsilon_j + \rho/2 + \xi}{\varepsilon_j + \rho/2 - \xi} S_j^- S_k^+\right) + \\ &+ \frac{2(\alpha+\beta)(\varepsilon_j + \rho/2) - 2\xi}{(\varepsilon_j + \rho/2)^2 - \xi^2} S_j^z.\end{aligned}$$

Choose $(\alpha + \beta) = -\gamma\rho/2$. Then this expression reduces to (2.30) as $\rho \to \infty$.

Thus, the connections for the BAE (summarised in Figure 3.2) also hold on the level of the conserved operators.

Remark 3.3. *For Richardson–Gaudin models an important object is the sum of the conserved operators. Since both* τ_j^{rat} (3.19) *and* τ_j^{trig} (3.16) *turn out to be equivalent to* τ_j

(2.26), *the sums* $\sum_{j=1}^{\mathcal{L}} \tau_j^{rat}$ *and* $\sum_{j=1}^{\mathcal{L}} \tau_j^{trig}$ *are equivalent to* $\sum_{j=1}^{\mathcal{L}} \tau_j = -2\gamma \sum_{j=1}^{\mathcal{L}} S_j^z$.

3.3 Eigenvalues of the conserved operators

We have shown, in the quasi-classical limit, the explicit connections between the BAE and conserved operators associated with the rational limit of the BQISM for Richardson–Gaudin systems, and the corresponding twisted-periodic trigonometric systems (Figure 3.2). We can also verify analogous connections between the eigenvalues of the conserved operators. While this necessarily follows from the equivalence of the conserved operators, it is useful as a consistency check as well as having the potential to provide some alternative insights into the methods used.

The eigenvalues λ_j in the quasi-classical limit are constructed as follows from (2.47) (similarly to the periodic case in Section 2.3.3):

$$\lim_{u\to\varepsilon_j}(u-\varepsilon_j)\Lambda(u) = \eta^2\lambda_j + \mathcal{O}(\eta^3).$$

Let us substitute the expression (2.47) and compute the limits of all the components separately. Let us start with $\tilde{a}(u)$ and $d(u)$ given by (2.49). We have

$$\begin{aligned}
&\lim_{u\to\varepsilon_j}(u-\varepsilon_j)\prod_{l=1}^{\mathcal{L}}\frac{\sinh(u-\varepsilon_l-\eta/2)\sinh(u+\varepsilon_l-\eta/2)}{\sinh(u-\varepsilon_l)\sinh(u+\varepsilon_l)} = \\
&\qquad = -\frac{\eta}{2}+\frac{\eta^2}{4}\left[\coth(2\varepsilon_j)+\sum_{k\neq j}^{\mathcal{L}}\big(\coth(\varepsilon_j-\varepsilon_k)+\coth(\varepsilon_j+\varepsilon_k)\big)\right]+\mathcal{O}(\eta^3),\\
&\lim_{u\to\varepsilon_j}(u-\varepsilon_j)\prod_{l=1}^{\mathcal{L}}\frac{\sinh(u-\varepsilon_l+\eta/2)\sinh(u+\varepsilon_l+\eta/2)}{\sinh(u-\varepsilon_l)\sinh(u+\varepsilon_l)} = \\
&\qquad = \frac{\eta}{2}+\frac{\eta^2}{4}\left[\coth(2\varepsilon_j)+\sum_{k\neq j}^{\mathcal{L}}\big(\coth(\varepsilon_j-\varepsilon_k)+\coth(\varepsilon_j+\varepsilon_k)\big)\right]+\mathcal{O}(\eta^3),
\end{aligned}$$

$$\begin{aligned}
&\sinh(2\varepsilon_j-\eta)\sinh\big(-\xi+\varepsilon_j+\eta(\beta+1/2)\big) = \\
&\quad = \big(\sinh(2\varepsilon_j)-\eta\cosh(2\varepsilon_j)\big)\big(-\sinh(\xi-\varepsilon_j)+\eta(\beta+1/2)\cosh(\xi-\varepsilon_j)\big)+\mathcal{O}(\eta^2) = \\
&\quad = -\sinh(2\varepsilon_j)\sinh(\xi-\varepsilon_j)+\eta(\beta+1/2)\sinh(2\varepsilon_j)\cosh(\xi-\varepsilon_j)+ \\
&\qquad +\eta\cosh(2\varepsilon_j)\sinh(\xi-\varepsilon_j)+\mathcal{O}(\eta^2),
\end{aligned}$$

$$\begin{aligned}
&\sinh(2\varepsilon_j+\eta)\sinh\big(-\xi-\varepsilon_j+\eta(\beta+1/2)\big)=\\
&\quad=\big(\sinh(2\varepsilon_j)+\eta\cosh(2\varepsilon_j)\big)\big(-\sinh(\xi+\varepsilon_j)+\eta(\beta+1/2)\cosh(\xi+\varepsilon_j)\big)+\mathcal{O}(\eta^2)=\\
&\quad=-\sinh(2\varepsilon_j)\sinh(\xi+\varepsilon_j)+\eta(\beta+1/2)\sinh(2\varepsilon_j)\cosh(\xi+\varepsilon_j)-\\
&\qquad-\eta\cosh(2\varepsilon_j)\sinh(\xi+\varepsilon_j)+\mathcal{O}(\eta^2).
\end{aligned}$$

Then we have

$$\begin{aligned}
&\lim_{u\to\varepsilon_j}(u-\varepsilon_j)\tilde{a}(u)=\\
&\quad=\Big[-\sinh(2\varepsilon_j)\sinh(\xi-\varepsilon_j)+\eta(\beta+1/2)\sinh(2\varepsilon_j)\cosh(\xi-\varepsilon_j)+\\
&\qquad+\eta\cosh(2\varepsilon_j)\sinh(\xi-\varepsilon_j)+\mathcal{O}(\eta^2)\Big]\times\\
&\qquad\times\left[-\frac{\eta}{2}+\frac{\eta^2}{4}\left(\coth(2\varepsilon_j)+\sum_{k\neq j}^{\mathcal{L}}\big(\coth(\varepsilon_j-\varepsilon_k)+\coth(\varepsilon_j+\varepsilon_k)\big)\right)+\mathcal{O}(\eta^3)\right]=\\
&\quad=\frac{\eta}{2}\sinh(2\varepsilon_j)\sinh(\xi-\varepsilon_j)-\frac{\eta^2}{2}(\beta+1/2)\sinh(2\varepsilon_j)\cosh(\xi-\varepsilon_j)-\\
&\qquad-\frac{\eta^2}{2}\cosh(2\varepsilon_j)\sinh(\xi-\varepsilon_j)-\frac{\eta^2}{4}\cosh(2\varepsilon_j)\sinh(\xi-\varepsilon_j)-\\
&\qquad-\frac{\eta^2}{4}\sinh(2\varepsilon_j)\sinh(\xi-\varepsilon_j)\sum_{k\neq j}^{\mathcal{L}}\big(\coth(\varepsilon_j-\varepsilon_k)+\coth(\varepsilon_j+\varepsilon_k)\big)+\mathcal{O}(\eta^3)
\end{aligned}$$

and

$$\begin{aligned}
&\sinh(2\varepsilon_j+\eta)\lim_{u\to\varepsilon_j}(u-\varepsilon_j)d(u)=\\
&\quad=\Big[-\sinh(2\varepsilon_j)\sinh(\xi+\varepsilon_j)+\eta(\beta+1/2)\sinh(2\varepsilon_j)\cosh(\xi+\varepsilon_j)-\\
&\qquad-\eta\cosh(2\varepsilon_j)\sinh(\xi+\varepsilon_j)+\mathcal{O}(\eta^2)\Big]\times\\
&\qquad\times\left[\frac{\eta}{2}+\frac{\eta^2}{4}\left(\coth(2\varepsilon_j)+\sum_{k\neq j}^{\mathcal{L}}\big(\coth(\varepsilon_j-\varepsilon_k)+\coth(\varepsilon_j+\varepsilon_k)\big)\right)+\mathcal{O}(\eta^3)\right]=\\
&\quad=-\frac{\eta}{2}\sinh(2\varepsilon_j)\sinh(\xi+\varepsilon_j)+\frac{\eta^2}{2}(\beta+1/2)\sinh(2\varepsilon_j)\cosh(\xi+\varepsilon_j)-\\
&\qquad-\frac{\eta^2}{2}\cosh(2\varepsilon_j)\sinh(\xi+\varepsilon_j)-\frac{\eta^2}{4}\cosh(2\varepsilon_j)\sinh(\xi+\varepsilon_j)-\\
&\qquad-\frac{\eta^2}{4}\sinh(2\varepsilon_j)\sinh(\xi+\varepsilon_j)\sum_{k\neq j}^{\mathcal{L}}\big(\coth(\varepsilon_j-\varepsilon_k)+\coth(\varepsilon_j+\varepsilon_k)\big)+\mathcal{O}(\eta^3).
\end{aligned}$$

Consider

$$
\begin{aligned}
\frac{\sinh(\xi^{+}+\varepsilon_j+\eta/2)}{\sinh(2\varepsilon_j)} &= \frac{\sinh(\xi+\varepsilon_j+\eta(\alpha+1/2))}{\sinh(2\varepsilon_j)} = \\
&= \frac{\sinh(\xi+\varepsilon_j)+\eta(\alpha+1/2)\cosh(\xi+\varepsilon_j)}{\sinh(2\varepsilon_j)} + \mathcal{O}(\eta^2), \\
\frac{\sinh(\xi^{+}-\varepsilon_j+\eta/2)}{\sinh(2\varepsilon_j)} &= \frac{\sinh(\xi-\varepsilon_j+\eta(\alpha+1/2))}{\sinh(2\varepsilon_j)} = \\
&= \frac{\sinh(\xi-\varepsilon_j)+\eta(\alpha+1/2)\cosh(\xi-\varepsilon_j)}{\sinh(2\varepsilon_j)} + \mathcal{O}(\eta^2).
\end{aligned}
$$

Then

$$
\begin{aligned}
&\lim_{u\to\varepsilon_j}(u-\varepsilon_j)\Lambda(u) = \\
&\quad = \frac{\eta}{2}\Bigg[\sinh(2\varepsilon_j)\sinh(\xi-\varepsilon_j) - \eta(\beta+1/2)\sinh(2\varepsilon_j)\cosh(\xi-\varepsilon_j) - \\
&\qquad - \frac{3\eta}{2}\cosh(2\varepsilon_j)\sinh(\xi-\varepsilon_j) - \\
&\qquad - \frac{\eta}{2}\sinh(2\varepsilon_j)\sinh(\xi-\varepsilon_j)\sum_{k\neq j}^{\mathcal{L}}\big(\coth(\varepsilon_j-\varepsilon_k)+\coth(\varepsilon_j+\varepsilon_k)\big) + \mathcal{O}(\eta^2)\Bigg] \times \\
&\qquad \times \left[\frac{\sinh(\xi+\varepsilon_j)}{\sinh(2\varepsilon_j)} + \eta(\alpha+1/2)\frac{\cosh(\xi+\varepsilon_j)}{\sinh(2\varepsilon_j)} + \mathcal{O}(\eta^2)\right] \times \\
&\qquad \times \left[1+\eta\sum_{i=1}^{N}\big(\coth(\varepsilon_j-v_i)+\coth(\varepsilon_j+v_i)\big) + \mathcal{O}(\eta^2)\right] + \\
&\qquad + \frac{\eta}{2}\Bigg[-\sinh(2\varepsilon_j)\sinh(\xi+\varepsilon_j) + \eta(\beta+1/2)\sinh(2\varepsilon_j)\cosh(\xi+\varepsilon_j) - \\
&\qquad\quad - \frac{3\eta}{2}\cosh(2\varepsilon_j)\sinh(\xi+\varepsilon_j) - \\
&\qquad\quad - \frac{\eta}{2}\sinh(2\varepsilon_j)\sinh(\xi+\varepsilon_j)\sum_{k\neq j}^{\mathcal{L}}\big(\coth(\varepsilon_j-\varepsilon_k)+\coth(\varepsilon_j+\varepsilon_k)\big) + \mathcal{O}(\eta^2)\Bigg] \times \\
&\qquad \times \left[\frac{\sinh(\xi-\varepsilon_j)}{\sinh(2\varepsilon_j)} + \eta(\alpha+1/2)\frac{\cosh(\xi-\varepsilon_j)}{\sinh(2\varepsilon_j)} + \mathcal{O}(\eta^2)\right] \times \\
&\qquad \times \left[1-\eta\sum_{i=1}^{N}\big(\coth(\varepsilon_j-v_i)+\coth(\varepsilon_j+v_i)\big) + \mathcal{O}(\eta^2)\right] = \\
&\quad = \frac{\eta^2}{2}\Bigg[(\alpha+\beta+1)\big(\sinh(\xi-\varepsilon_j)\cosh(\xi+\varepsilon_j) - \sinh(\xi+\varepsilon_j)\cosh(\xi-\varepsilon_j)\big) -
\end{aligned}
$$

$$
\begin{aligned}
&- 3\coth(2\varepsilon_j)\sinh(\xi-\varepsilon_j)\sinh(\xi+\varepsilon_j) + \\
&+ 2\sinh(\xi-\varepsilon_j)\sinh(\xi+\varepsilon_j)\sum_{i=1}^{N}\big(\coth(\varepsilon_j - v_i)+\coth(\varepsilon_j+v_i)\big) - \\
&- \sinh(\xi-\varepsilon_j)\sinh(\xi+\varepsilon_j)\sum_{k\neq j}^{\mathcal{L}}\big(\coth(\varepsilon_j-\varepsilon_k)+\coth(\varepsilon_j+\varepsilon_k)\big)\Bigg] + \mathcal{O}(\eta^3).
\end{aligned}
$$

It gives the eigenvalues for **Trig. BQISM** up to a factor of $\dfrac{1}{\sinh(\varepsilon_j+\xi)\sinh(\varepsilon_j-\xi)}$ as follows:

$$
\begin{aligned}
\lambda_j^{trig} = {} & \frac{\delta}{2}\big(\coth(\varepsilon_j-\xi)+\coth(\varepsilon_j+\xi)\big) + \frac{3}{2}\coth(2\varepsilon_j) + \\
& + \frac{1}{2}\sum_{k\neq j}^{\mathcal{L}}\big(\coth(\varepsilon_j-\varepsilon_k)+\coth(\varepsilon_j+\varepsilon_k)\big) - \\
& - \sum_{i=1}^{N}\big(\coth(\varepsilon_j-v_i)+\coth(\varepsilon_j+v_i)\big),
\end{aligned}
\tag{3.22}
$$

where $\delta = -(\alpha+\beta+1)$.

We can check that the constant terms agree in the eigenstates and the eigenvalues. To do this, we need to check that the action of τ_j^{trig} on the state $\Omega = \begin{pmatrix}0\\1\end{pmatrix}^{\otimes\mathcal{L}}$ is equal to the constant term in (3.22). Namely, that

$$
\begin{aligned}
\tau_j^{trig}\Omega = {} & \Bigg(\frac{1}{2}\sum_{k\neq j}^{\mathcal{L}}\big(\coth(\varepsilon_j-\varepsilon_k)+\coth(\varepsilon_j+\varepsilon_k)\big) + \frac{1}{2}\coth(2\varepsilon_j) - \frac{1}{\sinh(2\varepsilon_j)}\frac{\sinh(\varepsilon_j+\xi)}{\sinh(\varepsilon_j-\xi)} - \\
& - \frac{1}{2}\frac{(\alpha+\beta)\sinh(2\varepsilon_j)}{\sinh(\varepsilon_j+\xi)\sinh(\varepsilon_j-\xi)} + \frac{1}{2}\frac{\sinh(2\xi)}{\sinh(\varepsilon_j+\xi)\sinh(\varepsilon_j-\xi)}\Bigg)\Omega = \\
= {} & \Bigg(-\frac{1}{2}(\alpha+\beta+1)\big(\coth(\varepsilon_j-\xi)+\coth(\varepsilon_j+\xi)\big) + \frac{3}{2}\coth(2\varepsilon_j) + \\
& + \frac{1}{2}\sum_{k\neq j}^{\mathcal{L}}\big(\coth(\varepsilon_j-\varepsilon_k)+\coth(\varepsilon_j+\varepsilon_k)\big)\Bigg)\Omega.
\end{aligned}
$$

Indeed, by making repeated use of the identity

$$
\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)
$$

and other similar identities for hyperbolic functions, we may easily check that

$$\coth(\varepsilon_j - \xi) + \coth(\varepsilon_j + \xi) = \frac{\sinh(2\varepsilon_j)}{\sinh(\varepsilon_j + \xi)\sinh(\varepsilon_j - \xi)}$$

and

$$\begin{aligned}\frac{1}{\sinh(2\varepsilon_j)}\frac{\sinh(\varepsilon_j + \xi)}{\sinh(\varepsilon_j - \xi)} + \frac{1}{2}\frac{\sinh(2\xi)}{\sinh(\varepsilon_j + \xi)\sinh(\varepsilon_j - \xi)} &= \\ = \coth(2\varepsilon_j) - \frac{1}{2}\frac{\sinh(2\varepsilon_j)}{\sinh(\varepsilon_j + \xi)\sinh(\varepsilon_j - \xi)}\,.\end{aligned}$$

Therefore, $\tau_j^{trig}\Omega = \lambda_j^{trig}\Omega$ with λ_j^{trig} given by equation (3.22).

3.3.1 Variable change #1

We can obtain **Trig. BQISM′** by applying the variable change #1 given in (3.5):

$$\begin{aligned}\lambda_j^{trig'} = &\frac{\delta}{2}\big(\coth(\varepsilon_j + \rho/2 - \xi) + \coth(\varepsilon_j + \rho/2 + \xi)\big) + \frac{3}{2}\coth(2\varepsilon_j + \rho) + \\ &+ \frac{1}{2}\sum_{k\neq j}^{\mathcal{L}}\big(\coth(\varepsilon_j - \varepsilon_k) + \coth(\varepsilon_j + \varepsilon_k + \rho)\big) - \\ &- \sum_{i=1}^{N}\big(\coth(\varepsilon_j - v_i) + \coth(\varepsilon_j + v_i + \rho)\big).\end{aligned} \tag{3.23}$$

3.3.2 Attenuated limit

Now, as $\rho \to \infty$ in **Trig. BQISM′** (3.23), we obtain **Trig. QISM** (2.28):

$$\begin{aligned}\lambda_j^{trig'} \to \delta + \frac{3}{2} + \frac{1}{2}\sum_{k\neq j}^{\mathcal{L}}\big(\coth(\varepsilon_j - \varepsilon_k) + 1\big) - \sum_{i=1}^{N}\big(\coth(\varepsilon_j - v_i) + 1\big) = \\ = \gamma + \frac{1}{2}\sum_{k\neq j}^{\mathcal{L}}\coth(\varepsilon_j - \varepsilon_k) - \sum_{i=1}^{N}\coth(\varepsilon_j - v_i),\end{aligned}$$

where $\gamma = -(\alpha + \beta + N - \mathcal{L}/2)$.

3.3.3 Rational limit

The rational limit of **Trig. BQISM** (3.22) gives **Rat. BQISM**:

$$\lambda_j^{rat} = \frac{\delta\varepsilon_j}{\varepsilon_j^2 - \xi^2} + \frac{3}{4\varepsilon_j} + \sum_{k\neq j}^{\mathcal{L}} \frac{\varepsilon_j}{\varepsilon_j^2 - \varepsilon_k^2} - \sum_{i=1}^{N} \frac{2\varepsilon_j}{\varepsilon_j^2 - v_i^2}. \tag{3.24}$$

Or, multiplied by ε_j:

$$\varepsilon_j\lambda_j^{rat} = \frac{\delta\varepsilon_j^2}{\varepsilon_j^2 - \xi^2} + \frac{3}{4} + \sum_{k\neq j}^{\mathcal{L}} \frac{\varepsilon_j^2}{\varepsilon_j^2 - \varepsilon_k^2} - \sum_{i=1}^{N} \frac{2\varepsilon_j^2}{\varepsilon_j^2 - v_i^2}. \tag{3.25}$$

3.3.4 Equivalence of the rational BQISM and the trigonometric QISM

Set $\xi = 0$ in **Rat. BQISM** (3.25):

$$\varepsilon_j\lambda_j^{rat}\big|_{\xi=0} = \delta + \frac{3}{4} + \sum_{k\neq j}^{\mathcal{L}} \frac{\varepsilon_j^2}{\varepsilon_j^2 - \varepsilon_k^2} - \sum_{i=1}^{N} \frac{2\varepsilon_j^2}{\varepsilon_j^2 - v_i^2}.$$

Using $\dfrac{\varepsilon_j^2}{\varepsilon_j^2 - \varepsilon_k^2} = \dfrac{1}{2}\left(\dfrac{\varepsilon_j^2 + \varepsilon_k^2}{\varepsilon_j^2 - \varepsilon_k^2} + 1\right)$ we obtain

$$\varepsilon_j\lambda_j^{rat}\big|_{\xi=0} = \delta + \frac{3}{4} + \frac{(\mathcal{L}-1)}{2} - N + \frac{1}{2}\sum_{k\neq j}^{\mathcal{L}} \frac{\varepsilon_j^2 + \varepsilon_k^2}{\varepsilon_j^2 - \varepsilon_k^2} - \sum_{i=1}^{N} \frac{\varepsilon_j^2 + v_i^2}{\varepsilon_j^2 - v_i^2}.$$

Making a change of variables $\varepsilon_j \mapsto \exp\varepsilon_j$, we obtain **Trig. QISM** (2.28) up to a constant term $-3/4$:

$$\varepsilon_j\lambda_j^{rat}\big|_{\xi=0} = -\left(\alpha + \beta + N - \frac{\mathcal{L}}{2}\right) - \frac{3}{4} + \frac{1}{2}\sum_{k\neq j}^{\mathcal{L}} \coth(\varepsilon_j - \varepsilon_k) - \sum_{i=1}^{N} \coth(\varepsilon_j - v_i).$$

Now, we want to turn **Trig. QISM** (2.28) back into **Rat. BQISM** (3.25). We start with **Trig. QISM** (2.28) (with a change of variables $\varepsilon_j = \ln z_j,\ v_i = \ln y_i$)

$$\lambda^{(1)} = \gamma + \frac{1}{2}\sum_{k\neq j}^{\mathcal{L}} \frac{z_j^2 + z_k^2}{z_j^2 - z_k^2} - \sum_{i=1}^{N} \frac{z_j^2 + y_i^2}{z_j^2 - y_i^2} =$$
$$= \gamma + N - \frac{\mathcal{L}}{2} + \frac{1}{2} + \sum_{k\neq j}^{\mathcal{L}} \frac{z_j^2}{z_j^2 - z_k^2} - 2\sum_{i=1}^{N} \frac{z_j^2}{z_j^2 - y_i^2}.$$

Make the change of variables

$$z_j \mapsto \sqrt{\varepsilon_j^2 - \xi^2},\ y_i \mapsto \sqrt{v_i^2 - \xi^2}.$$

This gives

$$\lambda_j^{(2)} = \gamma + N - \frac{\mathcal{L}}{2} + \frac{1}{2} + \sum_{k\neq j}^{\mathcal{L}} \frac{\varepsilon_j^2 - \xi^2}{\varepsilon_j^2 - \varepsilon_k^2} - 2\sum_{i=1}^{N} \frac{\varepsilon_j^2 - \xi^2}{\varepsilon_j^2 - v_i^2}.$$

Then, rescale by $\dfrac{\varepsilon_j^2}{\varepsilon_j^2 - \xi^2}$:

$$\lambda_j^{(3)} = \left(\gamma + N - \frac{\mathcal{L}}{2} + \frac{1}{2}\right)\frac{\varepsilon_j^2}{\varepsilon_j^2 - \xi^2} + \sum_{k\neq j}^{\mathcal{L}} \frac{\varepsilon_j^2}{\varepsilon_j^2 - \varepsilon_k^2} - 2\sum_{i=1}^{N} \frac{\varepsilon_j^2}{\varepsilon_j^2 - v_i^2}.$$

Choose $\gamma = -(\alpha + \beta + N - \mathcal{L}/2)$, which leads to

$$\gamma + N - \frac{\mathcal{L}}{2} + \frac{1}{2} = -(\alpha + \beta) + \frac{1}{2} = -(\alpha + \beta + 1) + \frac{3}{2} = \delta + \frac{3}{2}.$$

Thus,

$$\lambda_j^{(3)} = \left(\delta + \frac{3}{2}\right)\frac{\varepsilon_j^2}{\varepsilon_j^2 - \xi^2} + \sum_{k\neq j}^{\mathcal{L}} \frac{\varepsilon_j^2}{\varepsilon_j^2 - \varepsilon_k^2} - 2\sum_{i=1}^{N} \frac{\varepsilon_j^2}{\varepsilon_j^2 - v_i^2}$$

is the same as **Rat. BQISM** (3.25) up to a constant term. Hence, **Trig. QISM** is equivalent to **Rat. BQISM** in the quasi-classical limit also on the level of the eigenvalue formula. The difference of the constants in the eigenvalues

$$\lambda_j^{(3)} - \varepsilon_j \lambda_j^{rat} = \frac{3}{2}\frac{\varepsilon_j^2}{\varepsilon_j^2 - \xi^2} - \frac{3}{4} = \frac{3}{4}\frac{\varepsilon_j^2 + \xi^2}{\varepsilon_j^2 - \xi^2}$$

is the same as the action of the difference of the conserved operators on the reference state:

$$\tau_j^{(4)}\Omega - \varepsilon_j \tau_j^{rat}\Omega = \left(\frac{\varepsilon_j^2}{\varepsilon_j^2 - \xi^2}\frac{1}{2} + \frac{\varepsilon_j^2 + \xi^2}{\varepsilon_j^2 - \xi^2}\frac{1}{2} - \frac{1}{4} \right)\Omega = \left(\frac{3}{4}\frac{\varepsilon_j^2 + \xi^2}{\varepsilon_j^2 - \xi^2} \right)\Omega.$$

3.3.5 Variable change #2

Here we want to transform the eigenvalue formula **Rat. BQISM** (3.24) back into **Trig. BQISM** (3.22). We start with **Rat. BQISM** in the form (3.25), multiplied by ε_j:

$$\tilde{\lambda}^{(1)} = \varepsilon_j \lambda_j^{rat} = \frac{\delta \varepsilon_j^2}{\varepsilon_j^2 - \xi^2} + \frac{3}{4} + \sum_{k \neq j}^{\mathcal{L}} \frac{\varepsilon_j^2}{\varepsilon_j^2 - \varepsilon_k^2} - \sum_{i=1}^{N} \frac{2\varepsilon_j^2}{\varepsilon_j^2 - v_i^2}.$$

We follow similar steps as in the case of the conserved operators, without the basis transformation. Start with the change of variables

$$\varepsilon_j \mapsto \frac{z_j - z_j^{-1}}{2}, \quad v_i \mapsto \frac{y_i - y_i^{-1}}{2}, \quad \xi \mapsto \frac{\chi - \chi^{-1}}{2}.$$

This gives

$$\tilde{\lambda}^{(2)} = \frac{\delta (z_j - z_j^{-1})^2}{(z_j - z_j^{-1})^2 - (\chi - \chi^{-1})^2} + \frac{3}{4} + \sum_{k \neq j}^{\mathcal{L}} \frac{(z_j - z_j^{-1})^2}{(z_j - z_j^{-1})^2 - (z_k - z_k^{-1})^2} - \sum_{i=1}^{N} \frac{2(z_j - z_j^{-1})^2}{(z_j - z_j^{-1})^2 - (y_i - y_i^{-1})^2}.$$

Now rescale by $\dfrac{z_j^2 - z_j^{-2}}{(z_j - z_j^{-1})^2}$:

$$\tilde{\lambda}^{(3)} = \frac{\delta (z_j^2 - z_j^{-2})}{(z_j - z_j^{-1})^2 - (\chi - \chi^{-1})^2} + \frac{3}{4}\frac{z_j^2 - z_j^{-2}}{(z_j - z_j^{-1})^2} + \sum_{k \neq j}^{\mathcal{L}} \frac{z_j^2 - z_j^{-2}}{(z_j - z_j^{-1})^2 - (z_k - z_k^{-1})^2} - \sum_{i=1}^{N} \frac{2(z_j^2 - z_j^{-2})}{(z_j - z_j^{-1})^2 - (y_i - y_i^{-1})^2}.$$

Using the identity

$$\frac{(x^2 - x^{-2})}{(x - x^{-1})^2 - (y - y^{-1})^2} = \frac{1}{2}\left(\frac{x^2 + y^2}{x^2 - y^2} + \frac{x^2 y^2 + 1}{x^2 y^2 - 1} \right)$$

we obtain

$$\tilde{\lambda}^{(3)} = \frac{\delta}{2}\left(\frac{z_j^2+\chi^2}{z_j^2-\chi^2} + \frac{z_j^2\chi^2+1}{z_j^2\chi^2-1}\right) + \frac{3}{4}\frac{z_j^2-z_j^{-2}}{(z_j-z_j^{-1})^2} + \frac{1}{2}\sum_{k\neq j}^{\mathcal{L}}\left(\frac{z_j^2+z_k^2}{z_j^2-z_k^2} + \frac{z_j^2z_k^2+1}{z_j^2z_k^2-1}\right) - \\ - \sum_{i=1}^{N}\left(\frac{z_j^2+y_i^2}{z_j^2-y_i^2} + \frac{z_j^2y_i^2+1}{z_j^2y_i^2-1}\right).$$

This is the same, up to a constant term, as **Trig. BQISM** (3.22) with the variable change $\varepsilon_j = \ln z_j,\ v_i = \ln y_i,\ \xi = \ln\chi$:

$$\lambda^{trig} = \frac{\delta}{2}\left(\frac{z_j^2+\chi^2}{z_j^2-\chi^2} + \frac{z_j^2\chi^2+1}{z_j^2\chi^2-1}\right) + \frac{3}{2}\frac{z_j^4+1}{z_j^4-1} + \frac{1}{2}\sum_{k\neq j}^{\mathcal{L}}\left(\frac{z_j^2+z_k^2}{z_j^2-z_k^2} + \frac{z_j^2z_k^2+1}{z_j^2z_k^2-1}\right) - \\ - \sum_{i=1}^{N}\left(\frac{z_j^2+y_i^2}{z_j^2-y_i^2} + \frac{z_j^2y_i^2+1}{z_j^2y_i^2-1}\right).$$

We have

$$\tilde{\lambda}^{(3)} - \lambda^{trig} = \frac{3}{4}\frac{z_j^2-z_j^{-2}}{(z_j-z_j^{-1})^2} - \frac{3}{2}\frac{z_j^4+1}{z_j^4-1}. \tag{3.26}$$

To check that the constants match with the constants from the conserved operators we need to compare the expression (3.26) above with the action of $\tau_j^{(4)} - \tau_j^{trig}$ on Ω:

$$\left(\tau_j^{(4)} - \tau_j^{trig}\right)\Omega = \left(\frac{1}{4}\frac{z_j^2-z_j^{-2}}{(z_j-z_j^{-1})^2} - \frac{1}{2}\frac{(z_j-z_j^{-1})^2+(\chi-\chi^{-1})^2}{(z_j-z_j^{-1})^2}\frac{\chi^2(z_j^4-1)}{(z_j^2-\chi^2)(z_j^2\chi^2-1)} - \right. \\ \left. - \frac{1}{2}\frac{z_j^4+1}{z_j^4-1} + \frac{z_j^2}{z_j^4-1}\left(\frac{z_j^2-\chi^2}{\chi^2z_j^2-1} + \frac{\chi^2z_j^2-1}{z_j^2-\chi^2}\right)\right)\Omega.$$

This expression is equivalent to (3.26) provided the following identity holds:

$$\frac{z_j^4+1}{z_j^4-1} - \frac{1}{2}\frac{z_j^2-z_j^{-2}}{(z_j-z_j^{-1})^2} = \frac{1}{2}\frac{(z_j-z_j^{-1})^2+(\chi-\chi^{-1})^2}{(z_j-z_j^{-1})^2}\frac{\chi^2(z_j^4-1)}{(z_j^2-\chi^2)(z_j^2\chi^2-1)} - \\ - \frac{z_j^2}{z_j^4-1}\left(\frac{z_j^2-\chi^2}{\chi^2z_j^2-1} + \frac{\chi^2z_j^2-1}{z_j^2-\chi^2}\right). \tag{3.27}$$

Simplifying the left hand side of (3.27) we find

$$\frac{z_j^4+1}{z_j^4-1} - \frac{1}{2}\frac{z_j^2-z_j^{-2}}{(z_j-z_j^{-1})^2} = \frac{1}{2}\frac{z_j-z_j^{-1}}{z_j+z_j^{-1}}.$$

Modifying the right hand side of (3.27) yields

$$\frac{1}{2}\frac{(z_j-z_j^{-1})^2+(\chi-\chi^{-1})^2}{(z_j-z_j^{-1})^2}\frac{\chi^2(z_j^4-1)}{(z_j^2-\chi^2)(z_j^2\chi^2-1)}-\frac{z_j^2}{z_j^4-1}\left(\frac{z_j^2-\chi^2}{\chi^2z_j^2-1}+\frac{\chi^2z_j^2-1}{z_j^2-\chi^2}\right)=$$
$$=\frac{1}{2}\frac{(z_j-z_j^{-1})^2+(\chi-\chi^{-1})^2}{(z_j-z_j^{-1})}\frac{\chi^2z_j^2(z_j+z_j^{-1})}{(z_j^2-\chi^2)(z_j^2\chi^2-1)}-$$
$$-\frac{1}{(z_j-z_j^{-1})(z_j+z_j^{-1})}\frac{(z_j^2-\chi^2)^2+(z_j^2\chi^2-1)^2}{(z_j^2-\chi^2)(z_j^2\chi^2-1)}=$$
$$=\frac{1}{2}\frac{(z_j+z_j^{-1})}{(z_j-z_j^{-1})}\frac{\chi^2(z_j^2-1)^2+z_j^2(\chi^2-1)^2}{(z_j^2-\chi^2)(z_j^2\chi^2-1)}-$$
$$-\frac{1}{(z_j-z_j^{-1})(z_j+z_j^{-1})}\frac{(z_j^2-\chi^2)^2+(z_j^2\chi^2-1)^2}{(z_j^2-\chi^2)(z_j^2\chi^2-1)}=$$
$$=\frac{(z_j+z_j^{-1})^2\left(\chi^2(z_j^2-1)^2+z_j^2(\chi^2-1)^2\right)-2(z_j^2-\chi^2)^2-2(z_j^2\chi^2-1)^2}{2(z_j-z_j^{-1})(z_j+z_j^{-1})(z_j^2-\chi^2)(z_j^2\chi^2-1)}=$$
$$=\frac{(z_j^2+z_j^{-2}+2)\chi^2(z_j^2-1)^2+(z_j^2+z_j^{-2}+2)z_j^2(\chi^2-1)^2-2(z_j^2-\chi^2)^2-2(z_j^2\chi^2-1)^2}{2(z_j-z_j^{-1})(z_j+z_j^{-1})(z_j^2-\chi^2)(z_j^2\chi^2-1)}.$$

The numerator above can be manipulated as

$$(z_j^4+1)\chi^2(z_j-z_j^{-1})^2+(z_j^4+1)(\chi^2-1)^2+$$
$$+2\Big[\chi^2(z_j^2-1)^2+z_j^2(\chi^2-1)^2-(z_j^2-\chi^2)^2-(z_j^2\chi^2-1)^2\Big]=$$
$$=(z_j^4+1)\chi^2(z_j-z_j^{-1})^2+(z_j^4+1)(\chi^2-1)^2+$$
$$+2\Big[\chi^2z_j^4+\chi^2+z_j^2\chi^4+z_j^2-z_j^4-\chi^4-\chi^4z_j^4-1\Big]=$$
$$=(z_j^4+1)\chi^2(z_j-z_j^{-1})^2+(z_j^4+1)(\chi^2-1)^2-$$
$$-2\chi^2z_j^4(\chi^2-1)+2\chi^4(z_j^2-1)-2z_j^2(z_j^2-1)+2(\chi^2-1)=$$
$$=(z_j^4+1)\chi^2(z_j-z_j^{-1})^2+2\chi^4z_j(z_j-z_j^{-1})-2z_j^3(z_j-z_j^{-1})+$$
$$+(\chi^2-1)\left[(z_j^4+1)(\chi^2-1)-2\chi^2z_j^4+2\right]=$$
$$=(z_j^4+1)\chi^2(z_j-z_j^{-1})^2+2(\chi^4-z_j^2)z_j(z_j-z_j^{-1})-(\chi^2-1)(\chi^2+1)(z_j^4-1)=$$
$$=(z_j-z_j^{-1})\Big[(z_j^4+1)\chi^2(z_j-z_j^{-1})+2z_j(\chi^4-z_j^2)-(\chi^4-1)z_j(z_j^2+1)\Big]=$$
$$=(z_j-z_j^{-1})\Big[(z_j^4+1)\chi^2(z_j-z_j^{-1})-(\chi^4+1)z_j(z_j^2-1)\Big]=$$
$$=(z_j-z_j^{-1})^2\Big[\chi^2(z_j^4+1)-z_j^2(\chi^4+1)\Big]=$$
$$=(z_j-z_j^{-1})^2(z_j^2-\chi^2)(z_j^2\chi^2-1).$$

Thus, the right hand side of (3.27) is

$$\frac{(z_j - z_j^{-1})^2(z_j^2 - \chi^2)(z_j^2\chi^2 - 1)}{2(z_j - z_j^{-1})(z_j + z_j^{-1})(z_j^2 - \chi^2)(z_j^2\chi^2 - 1)} = \frac{1}{2}\frac{z_j - z_j^{-1}}{z_j + z_j^{-1}},$$

verifying that (3.27) holds.

3.3.6 Variable change #3

The variable change 3 is obtained in the same way as for the BAE and conserved operators, described in Sections 3.1 and 3.2.

3.3.7 Reduction to the rational, twisted-periodic case

The rational limit of **Trig. QISM** (2.28) gives (2.31)

$$\lambda_j = \gamma + \frac{1}{2}\sum_{k\neq j}^{\mathcal{L}} \frac{1}{\varepsilon_j - \varepsilon_k} - \sum_{i=1}^{N} \frac{1}{\varepsilon_j - v_i}.$$

The rational limit of **Trig. BQISM′** gives **Rat. BQISM′**

$$\begin{aligned}\lambda_j^{rat'} = \frac{\delta(\varepsilon_j + \rho/2)}{(\varepsilon_j + \rho/2)^2 - \xi^2} + \frac{3}{2}\frac{1}{(2\varepsilon_j + \rho)} + \sum_{k\neq j}^{\mathcal{L}} \frac{\varepsilon_j + \rho/2}{(\varepsilon_j + \rho/2)^2 - (\varepsilon_k + \rho/2)^2} - \\ - \sum_{i=1}^{N} \frac{2(\varepsilon_j + \rho/2)}{(\varepsilon_j + \rho/2)^2 - (v_i + \rho/2)^2}.\end{aligned} \tag{3.28}$$

Choose $\delta = \rho\gamma/2$. Then we see that, as $\rho \to \infty$, (3.28) turns into (2.31).

Thus, the connections illustrated in Figure 3.2 also hold for the eigenvalues of the conserved operators.

3.4 Summary

In this chapter we have studied the spin-1/2 Richardson–Gaudin system as the quasi-classical limit of a generalised BQISM construction. In this manner we uncovered some surprising features, in particular, that the boundary trigonometric system is equivalent to its rational limit. Additionally we found that the twisted-periodic and boundary constructions are equivalent in the trigonometric case, but not in the rational limit. One consequence of this finding is that for the spin-1/2 Richardson–Gaudin system the BQISM formalism does not extend the integrable structure beyond that provided by the QISM formalism. This is an unexpected result, in contrast to the Heisenberg model.

Chapter 4

Rational Richardson–Gaudin models with off-diagonal reflection matrices

In this chapter we study Richardson–Gaudin model obtained in the quasi-classical limit of the BQISM construction with rational off-diagonal K-matrices. We construct a family of mutually commuting conserved operators and show how they lead to an integrable generalisation of the $p+ip$ Hamiltonian allowing for interaction with the environment, thus, giving a physical interpretation of the constructed model.

First of all, let us specify the main ingredients of our construction. One can check that the following K-matrix[1] satisfies the first reflection equation (2.32a) together with the rational R-matrix (2.3):

$$\check{K}^-(u) = \begin{pmatrix} \xi^- + u & \psi^- u \\ \phi^- u & \xi^- - u \end{pmatrix}.$$

Then, $\check{K}^+(u) = -\check{K}^-(-u-\eta)\big|_{\xi^-\mapsto-\xi^+,\psi^-\mapsto\psi^+,\phi^-\mapsto\phi^+}$ automatically satisfies the dual reflection equation (2.32b). Thus,

$$\check{K}^+(u) = \begin{pmatrix} \xi^+ + u + \eta & \psi^+(u+\eta) \\ \phi^+(u+\eta) & \xi^+ - u - \eta \end{pmatrix}.$$

As before (see Section 2.4) it is convenient to make a variable change $u \mapsto u - \eta/2,\ \varepsilon_j \mapsto$

[1]We will not introduce new notation for the K-matrices in each case, but we will specify them in the beginning of each chapter.

$\varepsilon_j - \eta/2$ and re-define all functions taking this into account:

$$K^-(u) = \begin{pmatrix} \xi^- + u - \eta/2 & \psi^-(u - \eta/2) \\ \phi^-(u - \eta/2) & \xi^- - u + \eta/2 \end{pmatrix}, \tag{4.1a}$$

$$K^+(u) = \begin{pmatrix} \xi^+ + u + \eta/2 & \psi^+(u + \eta/2) \\ \phi^+(u + \eta/2) & \xi^+ - u - \eta/2 \end{pmatrix}, \tag{4.1b}$$

The transfer matrix in this case is (2.44) with K-matrices given by (4.1) and the rational Lax operator (2.18):

$$t(u) = \operatorname{tr}_a \big(K_a^+(u) L_{a\mathcal{L}}(u - \varepsilon_{\mathcal{L}}) \cdots L_{a1}(u - \varepsilon_1) K_a^-(u) L_{a1}(u + \varepsilon_1) \cdots L_{a\mathcal{L}}(u + \varepsilon_{\mathcal{L}})\big). \tag{4.2}$$

The plan for this chapter is the following. First of all, we discuss how the attenuated limit works in this case. Then we construct the conserved operators in the quasi-classical limit (see expression (4.8) below) and prove that, like in the diagonal case, the second family of the conserved operators is equivalent to the first one (see Proposition 4.4 below). Next, using the fact that the rational Lax operator is invariant under local basis transformations, we bring one of the K-matrices to the diagonal form and simplify the expression for conserved operators to (4.10). In Section 4.2.4 we show how to construct a generalisation of the $p + ip$ Hamiltonian as a linear combination of these conserved operators, which includes extra interaction terms. In Section 4.2.5 we discuss a physical interpretation of these extra terms as interaction of the system with its environment. Finally, in Section 4.3, with help of the recently developed off-diagonal Bethe Ansatz method [WYCS15], we calculate the spectrum of the Hamiltonian subject to the corresponding BAE.

4.1 Attenuated limit

Let us investigate the attenuated limit of the transfer matrix (4.2) above. Firstly, note that the rational Lax operator (2.18) $L_{al}(u) \to I_a$ as $u \to \infty$. Let us rescale the K-matrices (4.1) in order to take the limit:

$$K^-(u) = \frac{1}{u} \begin{pmatrix} \xi^- + u - \eta/2 & \psi^-(u - \eta/2) \\ \phi^-(u - \eta/2) & \xi^- - u + \eta/2 \end{pmatrix},$$

$$K^+(u) = \frac{1}{u} \begin{pmatrix} \xi^+ + u + \eta/2 & \psi^+(u + \eta/2) \\ \phi^+(u + \eta/2) & \xi^+ - u - \eta/2 \end{pmatrix},$$

Then, in the limit as $u \to \infty$ we obtain

$$K^-(u) \xrightarrow{u\to\infty} \begin{pmatrix} 1 & \psi^- \\ \phi^- & -1 \end{pmatrix}, \; K^+(u) \xrightarrow{u\to\infty} \begin{pmatrix} 1 & \psi^+ \\ \phi^+ & -1 \end{pmatrix}.$$

Thus, from (4.2) we have

$$\begin{aligned} t(u) &\xrightarrow{\rho\to\infty} \mathrm{tr}_a \left(\begin{pmatrix} 1 & \psi^+ \\ \phi^+ & -1 \end{pmatrix}_a L_{a\mathcal{L}}(u-\varepsilon_{\mathcal{L}}) \cdots L_{a1}(u-\varepsilon_1) \begin{pmatrix} 1 & \psi^- \\ \phi^- & -1 \end{pmatrix}_a \right) = \\ &= \mathrm{tr}_a \left(\begin{pmatrix} 1 & \psi^- \\ \phi^- & -1 \end{pmatrix}_a \begin{pmatrix} 1 & \psi^+ \\ \phi^+ & -1 \end{pmatrix}_a L_{a\mathcal{L}}(u-\varepsilon_{\mathcal{L}}) \cdots L_{a1}(u-\varepsilon_1) \right) = \\ &= \mathrm{tr}_a \left(\begin{pmatrix} 1+\psi^-\phi^+ & \psi^+-\psi^- \\ \phi^- - \phi^+ & 1+\phi^-\psi^+ \end{pmatrix}_a L_{a\mathcal{L}}(u-\varepsilon_{\mathcal{L}}) \cdots L_{a1}(u-\varepsilon_1) \right). \end{aligned}$$

To obtain the twisted-periodic transfer matrix (2.9)

$$t(u) = \mathrm{tr}_a \left(\begin{pmatrix} e^{-\eta\gamma} & 0 \\ 0 & e^{\eta\gamma} \end{pmatrix}_a L_{a\mathcal{L}}(u-\varepsilon_{\mathcal{L}}) \cdots L_{a1}(u-\varepsilon_1) \right)$$

we need to transform $\tilde{M} = \begin{pmatrix} 1+\psi^-\phi^+ & \psi^+-\psi^- \\ \phi^- - \phi^+ & 1+\phi^-\psi^+ \end{pmatrix}$ into $M = \begin{pmatrix} e^{-\eta\gamma} & 0 \\ 0 & e^{\eta\gamma} \end{pmatrix}$. Note that the rational spin-1/2 Lax operator (2.18) is invariant under the local basis transformations:

$$X_a X_l L_{al}(u) X_a^{-1} X_l^{-1} = L_{al}(u)$$

for any invertible $X \in \mathrm{End}(\mathbb{C}^2)$. Thus, we can rewrite the transfer matrix as follows:

$$\begin{aligned} t(u) &= \mathrm{tr}_a \left(\tilde{M}_a L_{a\mathcal{L}}(u-\varepsilon_{\mathcal{L}}) \cdots L_{a1}(u-\varepsilon_1) \right) = \\ &= \mathrm{tr}_a \left(\tilde{M}_a X_a X_{\mathcal{L}} L_{a\mathcal{L}}(u-\varepsilon_{\mathcal{L}}) X_a^{-1} X_{\mathcal{L}}^{-1} \cdots X_a X_1 L_{a1}(u-\varepsilon_1) X_a^{-1} X_1^{-1} \right) = \\ &= X_{\mathcal{L}} \cdots X_1 \, \mathrm{tr}_a \left(\tilde{M}_a X_a L_{a\mathcal{L}}(u-\varepsilon_{\mathcal{L}}) \cdots L_{a1}(u-\varepsilon_1) X_a^{-1} \right) X_{\mathcal{L}}^{-1} \cdots X_1^{-1} = \\ &= X_{\mathcal{L}} \cdots X_1 \, \mathrm{tr}_a \left(X_a^{-1} \tilde{M}_a X_a L_{a\mathcal{L}}(u-\varepsilon_{\mathcal{L}}) \cdots L_{a1}(u-\varepsilon_1) \right) X_{\mathcal{L}}^{-1} \cdots X_1^{-1}. \end{aligned}$$

Finally, choose $X \in \mathrm{End}(V)$ so that $X^{-1}\tilde{M}X = M$, to match it with the twisted-periodic case.

4.2 Construction of conserved operators and the Hamiltonian

Unlike previous chapter where we started by considering the BAE, here we start directly by considering the quasi-classical limit of the transfer matrix[2]. We require (cf. Section 3.2.1) that the K-matrices (4.1) satisfy the condition

$$\lim_{\eta\to 0}\{K^+(u)K^-(u)\} \propto I, \tag{4.3}$$

which allows the quasi-classical expansion of the transfer matrix (4.2) to obtain conserved operators.

Assuming the following dependence of the parameters on η:

$$\begin{aligned} \xi^+ &= \xi + \eta\alpha, \quad \psi^+ = \psi + \eta\gamma, \quad \phi^+ = \phi + \eta\lambda, \\ \xi^- &= -\xi + \eta\beta, \quad \psi^- = \psi + \eta\delta, \quad \phi^- = \phi + \eta\mu, \end{aligned} \tag{4.4}$$

we can see that the condition (4.3) is satisfied:

$$K^+(u)K^-(u)\big|_{\eta=0} = \begin{pmatrix} \xi+u & \psi u \\ \phi u & \xi - u \end{pmatrix}\begin{pmatrix} -\xi+u & \psi u \\ \phi u & -\xi - u \end{pmatrix} = \big(u^2(1+\psi\phi) - \xi^2\big)I.$$

Now, expanding the K-matrices in η we obtain

$$K^+(u) = K_1^+(u) + \eta K_2^+(u) + \mathcal{O}(\eta^2) \tag{4.5}$$

with

$$K_1^+(u) = \begin{pmatrix} \xi+u & \psi u \\ \phi u & \xi - u \end{pmatrix}, \; K_2^+(u) = \begin{pmatrix} \alpha + 1/2 & \gamma u + \psi/2 \\ \lambda u + \phi/2 & \alpha - 1/2 \end{pmatrix},$$

and

$$K^-(u) = K_1^-(u) + \eta K_2^-(u) + \mathcal{O}(\eta^2) \tag{4.6}$$

with

$$K_1^-(u) = \begin{pmatrix} -\xi+u & \psi u \\ \phi u & -\xi - u \end{pmatrix}, \; K_2^-(u) = \begin{pmatrix} \beta - 1/2 & \delta u - \psi/2 \\ \mu u - \phi/2 & \beta + 1/2 \end{pmatrix}.$$

[2]Studying the BAE and the eigenvalues in the off-diagonal case requires more advanced techniques, which we will discuss in Section 4.3.

For the Lax operator (2.18) we have

$$L_{al}(u) = I + \frac{\eta}{u}\ell_{al} \text{ with } \ell_{al} = \begin{pmatrix} S_l^z & S_l^- \\ S_l^+ & -S_l^z \end{pmatrix}. \tag{4.7}$$

Remark 4.1. *Taking the quasi-classical limit of the reflection equations* (2.32) *gives the following relations between* $K_1^{\pm}(u)$ *and* $K_2^{\pm}(u)$*. From the* η*-term we obtain*

$$\frac{1}{u-v}\big(K_{b1}^-(u)K_{a1}^-(v) - K_{b1}^-(v)K_{a1}^-(u)\big) + \frac{1}{u+v}\big(K_{a1}^-(u)K_{a1}^-(v) - K_{b1}^-(v)K_{b1}^-(u)\big) = 0,$$

$$\frac{1}{u-v}\big(K_{b1}^+(u)K_{a1}^+(v) - K_{b1}^+(v)K_{a1}^+(u)\big) + \frac{1}{u+v}\big(K_{a1}^+(u)K_{a1}^+(v) - K_{b1}^+(v)K_{b1}^+(u)\big) = 0,$$

while from the η^2*-term we obtain*

$$\begin{aligned} &\frac{1}{u-v}\big(K_{b2}^-(u)K_{a1}^-(v) + K_{b1}^-(u)K_{a2}^-(v) - K_{b2}^-(v)K_{a1}^-(u) - K_{b1}^-(v)K_{a2}^-(u)\big) + \\ &+ \frac{1}{u+v}\big(K_{a2}^-(u)K_{a1}^-(v) + K_{a1}^-(u)K_{a2}^-(v) - K_{b2}^-(v)K_{b1}^-(u) - K_{b1}^-(v)K_{b2}^-(u)\big) = 0, \end{aligned}$$

$$\begin{aligned} &\frac{1}{u-v}\big(K_{b2}^+(u)K_{a1}^+(v) + K_{b1}^+(u)K_{a2}^+(v) - K_{b2}^+(v)K_{a1}^+(u) - K_{b1}^+(v)K_{a2}^+(u)\big) + \\ &+ \frac{1}{u+v}\big(K_{a2}^+(u)K_{a1}^+(v) + K_{a1}^+(u)K_{a2}^+(v) - K_{b2}^+(v)K_{b1}^+(u) - K_{b1}^+(v)K_{b2}^+(u)\big) = 0. \end{aligned}$$

Note that for this calculation we temporarily changed the indices (1,2) in the reflection equations (2.32) *to (a,b), so that they do not get confused with the indices (1,2) in the quasi-classical expansion of the* K*-matrices* (4.5) *and* (4.6)*.*

4.2.1 The first family of conserved operators

In the quasi-classical limit, the conserved operators τ_j[3] are constructed in the usual way from the transfer matrix (4.2) (same as (3.14) in previous chapter):

$$\lim_{u\to\varepsilon_j}(u-\varepsilon_j)t(u) = \eta^2\tau_j + \mathcal{O}(\eta^3).$$

[3]We will recycle the notation for the conserved operators and their eigenvalues.

Substituting (4.5), (4.6) and (4.7) into (4.2) we obtain

$$\lim_{u\to\varepsilon_j}(u-\varepsilon_j)t(u) = \eta\,\mathrm{tr}_a\Bigg[K^+_{1a}(\varepsilon_j)\ell_{aj}K^-_{1a}(\varepsilon_j) + \eta\sum_{k=j+1}^{\mathcal{L}}\frac{K^+_{1a}(\varepsilon_j)\ell_{ak}\ell_{aj}K^-_{1a}(\varepsilon_j)}{\varepsilon_j-\varepsilon_k} + \\ + \eta\sum_{k=1}^{j-1}\frac{K^+_{1a}(\varepsilon_j)\ell_{aj}\ell_{ak}K^-_{1a}(\varepsilon_j)}{\varepsilon_j-\varepsilon_k} + \eta K^+_{2a}(\varepsilon_j)\ell_{aj}K^-_{1a}(\varepsilon_j) + \\ + \eta\sum_{k=1}^{\mathcal{L}}\frac{K^+_{1a}(\varepsilon_j)\ell_{aj}K^-_{1a}(\varepsilon_j)\ell_{ak}}{\varepsilon_j+\varepsilon_k} + \eta K^+_{1a}(\varepsilon_j)\ell_{aj}K^-_{2a}(\varepsilon_j)\Bigg] + \mathcal{O}(\eta^3).$$

One can check that

$$\mathrm{tr}_a\left(K^+_{1a}(\varepsilon_j)\ell_{aj}K^-_{1a}(\varepsilon_j)\right) = 0,$$
$$\mathrm{tr}_a\left(K^+_{1a}(\varepsilon_j)\ell_{ak}\ell_{aj}K^-_{1a}(\varepsilon_j)\right) = \mathrm{tr}_a\left(K^+_{1a}(\varepsilon_j)\ell_{aj}\ell_{ak}K^-_{1a}(\varepsilon_j)\right).$$

Thus, we have

$$\tau_j = \sum_{k\neq j}^{\mathcal{L}}\frac{\mathrm{tr}_a\left(K^+_{1a}(\varepsilon_j)\ell_{ak}\ell_{aj}K^-_{1a}(\varepsilon_j)\right)}{\varepsilon_j-\varepsilon_k} + \sum_{k=1}^{\mathcal{L}}\frac{\mathrm{tr}_a\left(K^+_{1a}(\varepsilon_j)\ell_{aj}K^-_{1a}(\varepsilon_j)\ell_{ak}\right)}{\varepsilon_j+\varepsilon_k} + \\ + \mathrm{tr}_a\left(K^+_{2a}(\varepsilon_j)\ell_{aj}K^-_{1a}(\varepsilon_j)\right) + \mathrm{tr}_a\left(K^+_{1a}(\varepsilon_j)\ell_{aj}K^-_{2a}(\varepsilon_j)\right).$$

Computing the traces gives

$$\mathrm{tr}_a\left(K^+_{1a}(\varepsilon_j)\ell_{ak}\ell_{aj}K^-_{1a}(\varepsilon_j)\right) = \left((1+\psi\phi)\varepsilon_j^2-\xi^2\right)\left(2S^z_jS^z_k + S^+_jS^-_k + S^-_jS^+_k\right),$$
$$\mathrm{tr}_a\left(K^+_{1a}(\varepsilon_j)\ell_{aj}K^-_{1a}(\varepsilon_j)\ell_{ak}\right) = 2(\varepsilon_j+\xi)(\varepsilon_j-\xi)S^z_jS^z_k - (\varepsilon_j-\xi)^2S^+_jS^-_k - (\varepsilon_j+\xi)^2S^-_jS^+_k + \\ + 2\psi\varepsilon_j\left((\varepsilon_j+\xi)S^z_jS^+_k + (\varepsilon_j-\xi)S^+_jS^z_k\right) + \\ + 2\phi\varepsilon_j\left((\varepsilon_j+\xi)S^-_jS^z_k + (\varepsilon_j-\xi)S^z_jS^-_k\right) + \\ + \varepsilon_j^2\left(\psi^2S^+_jS^+_k + \phi^2S^-_jS^-_k - 2\psi\phi S^z_jS^z_k\right),$$
$$\mathrm{tr}_a\left(K^+_{2a}(\varepsilon_j)\ell_{aj}K^-_{1a}(\varepsilon_j)\right) = \left(2\alpha\varepsilon_j - \xi + (\lambda\psi-\gamma\phi)\varepsilon_j^2\right)S^z_j + \\ + \left((\alpha\psi-\gamma\xi)\varepsilon_j - \frac{\psi}{2}\xi + \gamma\varepsilon_j^2\right)S^+_j + \\ + \left((\alpha\phi-\lambda\xi)\varepsilon_j - \frac{\phi}{2}\xi - \lambda\varepsilon_j^2\right)S^-_j,$$
$$\mathrm{tr}_a\left(K^+_{1a}(\varepsilon_j)\ell_{aj}K^-_{2a}(\varepsilon_j)\right) = \left(2\beta\varepsilon_j - \xi + (\phi\delta-\psi\mu)\varepsilon_j^2\right)S^z_j + \\ + \left((\beta\psi+\delta\xi)\varepsilon_j - \frac{\psi}{2}\xi - \delta\varepsilon_j^2\right)S^+_j + \\ + \left((\beta\phi+\mu\xi)\varepsilon_j - \frac{\phi}{2}\xi + \mu\varepsilon_j^2\right)S^-_j.$$

Summing these up, we obtain a family of conserved operators for the rational Richardson–Gaudin model with off-diagonal boundary:

$$
\begin{aligned}
\tau_j = & \sum_{k\neq j}^{\mathcal{L}} \frac{(1+\psi\phi)\varepsilon_j^2-\xi^2}{\varepsilon_j-\varepsilon_k}\left(2S_j^zS_k^z+S_j^+S_k^-+S_j^-S_k^+\right)+ \\
& +\sum_{k=1}^{\mathcal{L}}\frac{1}{\varepsilon_j+\varepsilon_k}\Bigg(2(\varepsilon_j+\xi)(\varepsilon_j-\xi)S_j^zS_k^z-(\varepsilon_j-\xi)^2S_j^+S_k^- -(\varepsilon_j+\xi)^2S_j^-S_k^+ + \\
& \qquad +2\psi\varepsilon_j\left((\varepsilon_j+\xi)S_j^zS_k^+ +(\varepsilon_j-\xi)S_j^+S_k^z\right)+ \\
& \qquad +2\phi\varepsilon_j\left((\varepsilon_j+\xi)S_j^-S_k^z+(\varepsilon_j-\xi)S_j^zS_k^-\right)+ \\
& \qquad +\varepsilon_j^2\left(\psi^2S_j^+S_k^+ +\phi^2S_j^-S_k^- -2\psi\phi S_j^zS_k^z\right)\Bigg)+ \\
& +\left(2(\alpha+\beta)\varepsilon_j-2\xi+\psi(\lambda-\mu)\varepsilon_j^2-\phi(\gamma-\delta)\varepsilon_j^2\right)S_j^z+ \\
& +\left(\psi(\alpha+\beta)\varepsilon_j-\xi(\gamma-\delta)\varepsilon_j-\psi\xi+(\gamma-\delta)\varepsilon_j^2\right)S_j^+ + \\
& +\left(\phi(\alpha+\beta)\varepsilon_j-\xi(\lambda-\mu)\varepsilon_j-\phi\xi-(\lambda-\mu)\varepsilon_j^2\right)S_j^-.
\end{aligned}
\tag{4.8}
$$

Remark 4.2. *Assuming spin-1/2 representation we obtain the following expression for the sum of the conserved operators (cf. Remark 3.3 in Chapter 3 for the diagonal case):*

$$
\begin{aligned}
\sum_{j=1}^{\mathcal{L}}\tau_j = & \,2\sum_{j=1}^{\mathcal{L}}\sum_{k=j+1}^{\mathcal{L}}\frac{\varepsilon_j^2+\varepsilon_k^2-2\xi^2}{\varepsilon_j+\varepsilon_k}S_j^zS_k^z- \\
& -\sum_{j=1}^{\mathcal{L}}\sum_{k=j+1}^{\mathcal{L}}\frac{1}{\varepsilon_j+\varepsilon_k}\Big(\left((\varepsilon_j-\xi)^2+(\varepsilon_k+\xi)^2\right)S_j^+S_k^- +\left((\varepsilon_j+\xi)^2+(\varepsilon_k-\xi)^2\right)S_j^-S_k^+\Big)+ \\
& +2\sum_{j=1}^{\mathcal{L}}\sum_{k=j+1}^{\mathcal{L}}\frac{1}{\varepsilon_j+\varepsilon_k}\Big(\left(\varepsilon_j(\varepsilon_j+\xi)+\varepsilon_k(\varepsilon_k-\xi)\right)\left(\psi S_j^zS_k^+ +\phi S_j^-S_k^z\right)+ \\
& \qquad +\left(\varepsilon_j(\varepsilon_j-\xi)+\varepsilon_k(\varepsilon_k+\xi)\right)\left(\psi S_j^+S_k^z+\phi S_j^zS_k^-\right)\Big)+ \\
& +\sum_{j=1}^{\mathcal{L}}\sum_{k=j+1}^{\mathcal{L}}\frac{\varepsilon_j^2+\varepsilon_k^2}{\varepsilon_j+\varepsilon_k}\left(\psi^2S_j^+S_k^+ +\phi^2S_j^-S_k^- -2\psi\phi S_j^zS_k^z\right)+ \\
& +\sum_{j=1}^{\mathcal{L}}\left(2(\alpha+\beta)\varepsilon_j+\psi(\lambda-\mu)\varepsilon_j^2-\phi(\gamma-\delta)\varepsilon_j^2\right)S_j^z+ \\
& +\sum_{j=1}^{\mathcal{L}}\left(\psi(\alpha+\beta)\varepsilon_j-\xi(\gamma-\delta)\varepsilon_j+(\gamma-\delta)\varepsilon_j^2\right)S_j^+ + \\
& +\sum_{j=1}^{\mathcal{L}}\left(\phi(\alpha+\beta)\varepsilon_j-\xi(\lambda-\mu)\varepsilon_j-(\lambda-\mu)\varepsilon_j^2\right)S_j^-.
\end{aligned}
$$

Remark 4.3. *To obtain the diagonal limit we set $\phi = \psi = 0,\ \gamma = \delta = \lambda = \mu = 0$. Then (4.8) will reduce to*

$$\begin{aligned}\tau_j \to (\varepsilon_j - \xi)(\varepsilon_j + \xi)\Bigg[&\sum_{k\neq j}^{\mathcal{L}} \frac{1}{\varepsilon_j - \varepsilon_k}\left(2S_j^z S_k^z + S_j^+ S_k^- + S_j^- S_k^+\right) + \\ &+ \sum_{k=1}^{\mathcal{L}} \frac{1}{\varepsilon_j + \varepsilon_k}\left(2S_j^z S_k^z - \frac{\varepsilon_j - \xi}{\varepsilon_j + \xi} S_j^+ S_k^- - \frac{\varepsilon_j + \xi}{\varepsilon_j - \xi} S_j^- S_k^+\right) + \\ &+ \frac{2(\alpha + \beta)\varepsilon_j - 2\xi}{(\varepsilon_j - \xi)(\varepsilon_j + \xi)} S_j^z\Bigg] = \\ = (\varepsilon_j - \xi)(\varepsilon_j + \xi)\tau_j^{rat}.&\end{aligned}$$

Up to a scalar it is the same as (3.19).

4.2.2 The second family of conserved operators

Note that we have only considered one of two possible families of the conserved operators so far. The second family is constructed as follows from the transfer matrix (4.2) (same as (3.17) from Section 3.2.2):

$$\lim_{u\to-\varepsilon_j} (u + \varepsilon_j) t(u) = \eta^2 \tilde{\tau}_j + \mathcal{O}(\eta^3).$$

Also in this case we can formulate the following proposition:

Proposition 4.4. *The second family of conserved operators is equivalent to the first, in particular, $\tilde{\tau}_j = -\tau_j$.*

Proof. We start in the same way as in the proof of Proposition 3.2. First of all, denote

$$t(u, \vec{\varepsilon}) = \mathrm{tr}_a\Big(K_a^+(u) L_{a\mathcal{L}}(u - \varepsilon_{\mathcal{L}}) \cdots L_{a1}(u - \varepsilon_1) K_a^-(u) L_{a1}(u + \varepsilon_1) \cdots L_{a\mathcal{L}}(u + \varepsilon_{\mathcal{L}})\Big).$$

Then, consider $t(u, \vec{\varepsilon})^T$, where $T = t_1 \cdots t_{\mathcal{L}}$ denotes the transpose over all spaces. Using

$$(\mathrm{tr}_a A_a)^{t_1\cdots t_{\mathcal{L}}} = \mathrm{tr}_a\left(A_a^{t_1\cdots t_{\mathcal{L}}}\right) = \mathrm{tr}_a\left(A_a^{t_a t_1\cdots t_{\mathcal{L}}}\right),$$

the fact that the rational Lax operator is symmetric:

$$L_{al}(u)^T = \frac{1}{u}\begin{pmatrix} u + \eta(S_l^z)^T & \eta(S_l^+)^T \\ \eta(S_l^-)^T & u - \eta(S_l^z)^T \end{pmatrix} = \frac{1}{u}\begin{pmatrix} u + \eta S_l^z & \eta S_l^- \\ \eta S_l^+ & u - \eta S_l^z \end{pmatrix} = L_{al}(u),$$

and an observation that $K^+(u)^T = K^+(u)|_{\psi^+\mapsto\phi^+}$ and $K^-(u)^T = K^-(u)|_{\psi^-\mapsto\phi^-}$, we obtain

$$\begin{aligned} t(u,\vec{\varepsilon})^T &= \mathrm{tr}_a\left(L_{a\mathcal{L}}(u+\varepsilon_{\mathcal{L}})\cdots L_{a1}(u+\varepsilon_1)K_a^-(u)^T L_{a1}(u-\varepsilon_1)\cdots L_{a\mathcal{L}}(u-\varepsilon_N)K_a^+(u)^T\right) = \\ &= \mathrm{tr}_a\Big(K_a^+(u)L_{a\mathcal{L}}(u+\varepsilon_{\mathcal{L}})\cdots L_{a1}(u+\varepsilon_1)\times \\ &\qquad \times K_a^-(u)L_{a1}(u-\varepsilon_1)\cdots L_{a\mathcal{L}}(u-\varepsilon_{\mathcal{L}})\Big)\Big|_{\psi^+\mapsto\phi^+,\psi^-\mapsto\phi^-} = \\ &= t(u,-\vec{\varepsilon})|_{\psi^+\mapsto\phi^+,\psi^-\mapsto\phi^-}\,. \end{aligned}$$

Thus, we obtain the following equality:

$$t(u,\vec{\varepsilon}) = t(u,-\vec{\varepsilon})^T\Big|_{\psi^+\mapsto\phi^+,\psi^-\mapsto\phi^-}\,.$$

It follows that

$$\lim_{u\to-\varepsilon_j}(u+\varepsilon_j)t(u,\vec{\varepsilon}) = \lim_{u\to-\varepsilon_j}(u-(-\varepsilon_j))\,t(u,-\vec{\varepsilon})^T\Big|_{\psi^+\mapsto\phi^+,\psi^-\mapsto\phi^-}\,.$$

Thus, we have

$$\tilde{\tau}_j(\vec{\varepsilon}) = \tau_j(-\vec{\varepsilon})^T\Big|_{\psi^+\mapsto\phi^+,\psi^-\mapsto\phi^-} = \tau_j(-\vec{\varepsilon})^T\Big|_{\psi\mapsto\phi,\gamma\mapsto\lambda,\delta\mapsto\mu}\,.$$

In order to compute $\tilde{\tau}_j(\vec{\varepsilon}) = \tau_j(-\vec{\varepsilon})^T\Big|_{\psi\mapsto\phi,\gamma\mapsto\lambda,\delta\mapsto\mu}$ let us first rewrite the expression (4.8) in a more convenient form (separating the term with $k=j$ from the second sum):

$$\begin{aligned} \tau_j(\vec{\varepsilon}) = &\sum_{k\neq j}^{\mathcal{L}}\frac{(1+\psi\phi)\varepsilon_j^2-\xi^2}{\varepsilon_j-\varepsilon_k}\left(2S_j^zS_k^z+S_j^+S_k^-+S_j^-S_k^+\right)+ \\ &+\sum_{k\neq j}^{\mathcal{L}}\frac{1}{\varepsilon_j+\varepsilon_k}\Bigg(2(\varepsilon_j+\xi)(\varepsilon_j-\xi)S_j^zS_k^z-(\varepsilon_j-\xi)^2S_j^+S_k^--(\varepsilon_j+\xi)^2S_j^-S_k^++ \\ &\qquad+2\psi\varepsilon_j\left((\varepsilon_j+\xi)S_j^zS_k^++(\varepsilon_j-\xi)S_j^+S_k^z\right)+ \\ &\qquad+2\phi\varepsilon_j\left((\varepsilon_j+\xi)S_j^-S_k^z+(\varepsilon_j-\xi)S_j^zS_k^-\right)+ \\ &\qquad+\varepsilon_j^2\left(\psi^2S_j^+S_k^++\phi^2S_j^-S_k^--2\psi\phi S_j^zS_k^z\right)\Bigg)+ \\ &+\frac{1}{2\varepsilon_j}\Bigg(2(\varepsilon_j+\xi)(\varepsilon_j-\xi)(S_j^z)^2-(\varepsilon_j-\xi)^2S_j^+S_j^--(\varepsilon_j+\xi)^2S_j^-S_j^++ \\ &\qquad+2\psi\varepsilon_j\left((\varepsilon_j+\xi)S_j^zS_j^++(\varepsilon_j-\xi)S_j^+S_j^z\right)+ \\ &\qquad+2\phi\varepsilon_j\left((\varepsilon_j+\xi)S_j^-S_j^z+(\varepsilon_j-\xi)S_j^zS_j^-\right)+ \\ &\qquad+\varepsilon_j^2\left(\psi^2(S_j^+)^2+\phi^2(S_j^-)^2-2\psi\phi(S_j^z)^2\right)\Bigg)+ \end{aligned}$$

$$
\begin{aligned}
&+\left(2(\alpha+\beta)\varepsilon_j-2\xi+\psi(\lambda-\mu)\varepsilon_j^2-\phi(\gamma-\delta)\varepsilon_j^2\right)S_j^z+\\
&+\left(\psi(\alpha+\beta)\varepsilon_j-\xi(\gamma-\delta)\varepsilon_j-\psi\xi+(\gamma-\delta)\varepsilon_j^2\right)S_j^++\\
&+\left(\phi(\alpha+\beta)\varepsilon_j-\xi(\lambda-\mu)\varepsilon_j-\phi\xi-(\lambda-\mu)\varepsilon_j^2\right)S_j^-.
\end{aligned}
$$

Using the commutation relations $[S^z,S^+]=S^+$ and $[S^z,S^-]=-S^-$ we can simplify

$$
\begin{aligned}
\tau_j(\vec{\varepsilon})=&\sum_{k\neq j}^{\mathcal{L}}\frac{(1+\psi\phi)\varepsilon_j^2-\xi^2}{\varepsilon_j-\varepsilon_k}\left(2S_j^zS_k^z+S_j^+S_k^-+S_j^-S_k^+\right)+\\
&+\sum_{k\neq j}^{\mathcal{L}}\frac{1}{\varepsilon_j+\varepsilon_k}\Bigg(2(\varepsilon_j+\xi)(\varepsilon_j-\xi)S_j^zS_k^z-(\varepsilon_j-\xi)^2S_j^+S_k^--(\varepsilon_j+\xi)^2S_j^-S_k^++\\
&\qquad+2\psi\varepsilon_j\left((\varepsilon_j+\xi)S_j^zS_k^++(\varepsilon_j-\xi)S_j^+S_k^z\right)+\\
&\qquad+2\phi\varepsilon_j\left((\varepsilon_j+\xi)S_j^-S_k^z+(\varepsilon_j-\xi)S_j^zS_k^-\right)+\\
&\qquad+\varepsilon_j^2\left(\psi^2S_j^+S_k^++\phi^2S_j^-S_k^--2\psi\phi S_j^zS_k^z\right)\Bigg)+\\
&+\frac{1}{2\varepsilon_j}\Bigg(2(\varepsilon_j+\xi)(\varepsilon_j-\xi)(S_j^z)^2-(\varepsilon_j-\xi)^2S_j^+S_j^--(\varepsilon_j+\xi)^2S_j^-S_j^++\\
&\qquad+2\psi\varepsilon_j\left(\varepsilon_j(S_j^zS_j^++S_j^+S_j^z)+\xi S_j^+\right)+\\
&\qquad+2\phi\varepsilon_j\left(\varepsilon_j(S_j^zS_j^-+S_j^-S_j^z)+\xi S_j^-\right)+\\
&\qquad+\varepsilon_j^2\left(\psi^2(S_j^+)^2+\phi^2(S_j^-)^2-2\psi\phi(S_j^z)^2\right)\Bigg)+\\
&+\left(2(\alpha+\beta)\varepsilon_j-2\xi+\psi(\lambda-\mu)\varepsilon_j^2-\phi(\gamma-\delta)\varepsilon_j^2\right)S_j^z+\\
&+\left(\psi(\alpha+\beta)\varepsilon_j-\xi(\gamma-\delta)\varepsilon_j-\psi\xi+(\gamma-\delta)\varepsilon_j^2\right)S_j^++\\
&+\left(\phi(\alpha+\beta)\varepsilon_j-\xi(\lambda-\mu)\varepsilon_j-\phi\xi-(\lambda-\mu)\varepsilon_j^2\right)S_j^-.
\end{aligned}
$$

Now, one can compute $\tilde{\tau}_j(\vec{\varepsilon})=\tau_j(-\vec{\varepsilon})^T\big|_{\psi\mapsto\phi,\gamma\mapsto\lambda,\delta\mapsto\mu}$:

$$
\begin{aligned}
\tilde{\tau}_j(\vec{\varepsilon})=&-\sum_{k\neq j}^{\mathcal{L}}\frac{(1+\psi\phi)\varepsilon_j^2-\xi^2}{\varepsilon_j-\varepsilon_k}\left(2S_j^zS_k^z+S_j^+S_k^-+S_j^-S_k^+\right)-\\
&-\sum_{k=1}^{\mathcal{L}}\frac{1}{\varepsilon_j+\varepsilon_k}\Bigg(2(\varepsilon_j-\xi)(\varepsilon_j+\xi)S_j^zS_k^z-(\varepsilon_j+\xi)^2S_j^-S_k^+-(\varepsilon_j-\xi)^2S_j^+S_k^-+\\
&\qquad+2\phi\varepsilon_j\left((\varepsilon_j-\xi)S_j^zS_k^-+(\varepsilon_j+\xi)S_j^-S_k^z\right)+\\
&\qquad+2\psi\varepsilon_j\left((\varepsilon_j-\xi)S_j^+S_k^z+(\varepsilon_j+\xi)S_j^zS_k^+\right)+\\
&\qquad+\varepsilon_j^2\left(\psi^2S_j^+S_k^++\phi^2S_j^-S_k^--2\phi\psi S_j^zS_k^z\right)\Bigg)-
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2\varepsilon_j}\Bigg(2(\varepsilon_j-\xi)(\varepsilon_j+\xi)(S_j^z)^2-(\varepsilon_j+\xi)^2S_j^+S_j^- -(\varepsilon_j-\xi)^2S_j^-S_j^+ +\\
&\qquad +2\phi\varepsilon_j\left(\varepsilon_j(S_j^zS_j^-+S_j^-S_j^z)-\xi S_j^-\right)+\\
&\qquad +2\psi\varepsilon_j\left(\varepsilon_j(S_j^zS_j^++S_j^+S_j^z)-\xi S_j^+\right)+\\
&\qquad +\varepsilon_j^2\left(\phi^2(S_j^-)^2+\psi^2(S_j^+)^2-2\phi\psi(S_j^z)^2\right)\Bigg)+\\
&+\left(-2(\alpha+\beta)\varepsilon_j-2\xi+\phi(\gamma-\delta)\varepsilon_j^2-\psi(\lambda-\mu)\varepsilon_j^2\right)S_j^z+\\
&+\left(-\psi(\alpha+\beta)\varepsilon_j+\xi(\gamma-\delta)\varepsilon_j-\psi\xi-(\gamma-\delta)\varepsilon_j^2\right)S_j^++\\
&+\left(-\phi(\alpha+\beta)\varepsilon_j+\xi(\lambda-\mu)\varepsilon_j-\phi\xi+(\lambda-\mu)\varepsilon_j^2\right)S_j^-.
\end{aligned}
$$

After computing the sum

$$
\begin{aligned}
\tau_j(\vec{\varepsilon})+\tilde{\tau}_j(\vec{\varepsilon}) &= \frac{1}{2\varepsilon_j}\Big(\left((\varepsilon_j+\xi)^2-(\varepsilon_j-\xi)^2\right)[S_j^+,S_j^-]+4\psi\varepsilon_j\xi S_j^++4\phi\varepsilon_j\xi S_j^-\Big)-\\
&\quad -4\xi S_j^z-2\psi\xi S_j^+-2\phi\xi S_j^- =\\
&= \frac{1}{2\varepsilon_j}\Big(4\varepsilon_j\xi(2S_j^z)+4\psi\varepsilon_j\xi S_j^++4\phi\varepsilon_j\xi S_j^-\Big)-4\xi S_j^z-2\psi\xi S_j^+-2\phi\xi S_j^- = 0,
\end{aligned}
$$

we see that $\tilde{\tau}_j(\vec{\varepsilon}) = -\tau_j(\vec{\varepsilon})$. Thus, we have shown that the second family of the conserved operators is equivalent to the first one. □

Remark 4.5. *Note that we have not used the properties of spin-1/2 representation in Sections 4.2.1 and 4.2.2 (except for Remark 4.2). So, the expression (4.8) for the conserved operators and the equivalence of the two families of the conserved operators is true for arbitrary spin. In the following we will restrict to the case of spin-1/2 representation.*

4.2.3 The case when one K-matrix is diagonal

It now turns out that six of the parameters appearing in (4.8) are superfluous and can be eliminated by appropriate basis transformations and redefinitions of variables. First note that we can set $\beta = 0$ without loss of generality, since the dependence of (4.8) on α and β is only through the sum $\alpha+\beta$. Next, as we have already mentioned in Section 4.1 the rational spin-1/2 Lax operator (2.18) is invariant under local basis transformations, i.e.,

$$X_aX_lL_{al}(u)X_a^{-1}X_l^{-1} = L_{al}(u)$$

for any invertible $X \in \mathrm{End}(V)$. Thus, we can almost always choose a basis in which one of the K-matrices is diagonal. (The case when a K-matrix is not diagonalisable has been discussed in [AMS14]). For our purposes, we assume that $K^-(u)$ (4.1a) is diagonal, so that

$$K^-(u) = \begin{pmatrix} \xi^- + u - \eta/2 & 0 \\ 0 & \xi^- - u + \eta/2 \end{pmatrix},$$
$$K^+(u) = \begin{pmatrix} \xi^+ + u + \eta/2 & \psi^+(u+\eta/2) \\ \phi^+(u+\eta/2) & \xi^+ - u - \eta/2 \end{pmatrix}.$$

For the expansion (4.4) this means that $\psi = \phi = 0,\ \delta = \mu = 0$. Substituting these (together with $\beta = 0$) into (4.8) we obtain

$$\begin{aligned}
\tau_j = &\sum_{k\neq j}^{\mathcal{L}} \frac{(\varepsilon_j - \xi)(\varepsilon_j + \xi)}{\varepsilon_j - \varepsilon_k}\big(2S_j^z S_k^z + S_j^+ S_k^- + S_j^- S_k^+\big) + \\
&+ \sum_{k=1}^{\mathcal{L}} \frac{1}{\varepsilon_j + \varepsilon_k}\Big(2(\varepsilon_j+\xi)(\varepsilon_j-\xi)S_j^z S_k^z - (\varepsilon_j-\xi)^2 S_j^+ S_k^- - (\varepsilon_j+\xi)^2 S_j^- S_k^+\Big) + \\
&+ \big(2\alpha\varepsilon_j - 2\xi\big)S_j^z + \big(-\xi\gamma\varepsilon_j + \gamma\varepsilon_j^2\big)S_j^+ + \big(-\xi\lambda\varepsilon_j - \lambda\varepsilon_j^2\big)S_j^- = \\
= &(\varepsilon_j - \xi)(\varepsilon_j + \xi)\Bigg[\sum_{k\neq j}^{\mathcal{L}} \frac{1}{\varepsilon_j - \varepsilon_k}\big(2S_j^z S_k^z + S_j^+ S_k^- + S_j^- S_k^+\big) + \\
&\qquad + \sum_{k=1}^{\mathcal{L}} \frac{1}{\varepsilon_j + \varepsilon_k}\left(2S_j^z S_k^z - \frac{\varepsilon_j - \xi}{\varepsilon_j + \xi} S_j^+ S_j^- - \frac{\varepsilon_j + \xi}{\varepsilon_j - \xi} S_j^- S_k^+\right) + \\
&\qquad + \frac{2\alpha\varepsilon_j - 2\xi}{(\varepsilon_j - \xi)(\varepsilon_j + \xi)} S_j^z + \frac{\gamma\varepsilon_j}{\varepsilon_j + \xi} S_j^+ - \frac{\lambda\varepsilon_j}{\varepsilon_j - \xi} S_j^-\Bigg].
\end{aligned}$$

Rewrite it as follows:

$$\begin{aligned}
\tau_j = (\varepsilon_j - \xi)(\varepsilon_j + \xi)\Bigg[&\sum_{k\neq j}^{\mathcal{L}} \left(\frac{1}{\varepsilon_j - \varepsilon_k} + \frac{1}{\varepsilon_j + \varepsilon_k}\right) 2S_j^z S_k^z + \\
&+ \sum_{k\neq j}^{\mathcal{L}} \left(\frac{1}{\varepsilon_j - \varepsilon_k} - \frac{1}{\varepsilon_j + \varepsilon_k}\frac{\varepsilon_j - \xi}{\varepsilon_j + \xi}\right) S_j^+ S_k^- + \\
&+ \sum_{k\neq j}^{\mathcal{L}} \left(\frac{1}{\varepsilon_j - \varepsilon_k} - \frac{1}{\varepsilon_j + \varepsilon_k}\frac{\varepsilon_j + \xi}{\varepsilon_j - \xi}\right) S_j^- S_k^+ + \\
&+ \frac{1}{2\varepsilon_j} 2(S_j^z)^2 - \frac{1}{2\varepsilon_j}\frac{\varepsilon_j - \xi}{\varepsilon_j + \xi} S_j^+ S_j^- - \frac{1}{2\varepsilon_j}\frac{\varepsilon_j + \xi}{\varepsilon_j - \xi} S_j^- S_j^+ + \\
&+ \frac{2\alpha\varepsilon_j}{\varepsilon_j^2 - \xi^2} S_j^z - \frac{2\xi}{\varepsilon_j^2 - \xi^2} S_j^z + \frac{\gamma\varepsilon_j}{\varepsilon_j + \xi} S_j^+ - \frac{\lambda\varepsilon_j}{\varepsilon_j - \xi} S_j^-\Bigg].
\end{aligned}$$

Finally we may set $\xi = 0$ without loss of generality, although this is more technical to establish. Using the properties of the spin-1/2 representation, namely

$$S^+S^- = \frac{I}{2} + S^z,\ S^-S^+ = \frac{I}{2} - S^z,\ (S^z)^2 = \frac{I}{4},$$

and the identities

$$\frac{1}{\varepsilon_j - \varepsilon_k} - \frac{1}{\varepsilon_j + \varepsilon_k}\frac{\varepsilon_j - \xi}{\varepsilon_j + \xi} = \frac{2\varepsilon_j(\varepsilon_k + \xi)}{(\varepsilon_j^2 - \varepsilon_k^2)(\varepsilon_j + \xi)},$$
$$\frac{1}{\varepsilon_j - \varepsilon_k} - \frac{1}{\varepsilon_j + \varepsilon_k}\frac{\varepsilon_j + \xi}{\varepsilon_j - \xi} = \frac{2\varepsilon_j(\varepsilon_k - \xi)}{(\varepsilon_j^2 - \varepsilon_k^2)(\varepsilon_j - \xi)},$$

we obtain

$$\tau_j = (\varepsilon_j - \xi)(\varepsilon_j + \xi)\left[\sum_{k\neq j}^{\mathcal{L}} \frac{4\varepsilon_j}{\varepsilon_j^2 - \varepsilon_k^2} S_j^z S_k^z + \sum_{k\neq j}^{\mathcal{L}} \frac{2\varepsilon_j}{\varepsilon_j^2 - \varepsilon_k^2}\left(\frac{\varepsilon_k + \xi}{\varepsilon_j + \xi} S_j^+ S_k^- + \frac{\varepsilon_k - \xi}{\varepsilon_j - \xi} S_j^- S_k^+\right) + \right.$$
$$\left. + \frac{2\alpha\varepsilon_j}{\varepsilon_j^2 - \xi^2} S_j^z + \frac{\gamma\varepsilon_j}{\varepsilon_j + \xi} S_j^+ - \frac{\lambda\varepsilon_j}{\varepsilon_j - \xi} S_j^- + \frac{I}{4\varepsilon_j} - \frac{\varepsilon_j^2 + \xi^2}{\varepsilon_j^2 - \xi^2}\frac{I}{2\varepsilon_j}\right].$$

Rewrite it as follows:

$$\begin{aligned} \frac{\varepsilon_j \tau_j}{(\varepsilon_j - \xi)(\varepsilon_j + \xi)} = &\sum_{k\neq j}^{\mathcal{L}} \frac{4\varepsilon_j^2}{\varepsilon_j^2 - \varepsilon_k^2} S_j^z S_k^z + \\ &+ \sum_{k\neq j}^{\mathcal{L}} \frac{2\varepsilon_j^2}{\varepsilon_j^2 - \varepsilon_k^2}\left(\frac{\varepsilon_k + \xi}{\varepsilon_j + \xi} S_j^+ S_k^- + \frac{\varepsilon_k - \xi}{\varepsilon_j - \xi} S_j^- S_k^+\right) + \\ &+ \frac{2\alpha\varepsilon_j^2}{\varepsilon_j^2 - \xi^2} S_j^z + \frac{\gamma\varepsilon_j^2}{\varepsilon_j + \xi} S_j^+ - \frac{\lambda\varepsilon_j^2}{\varepsilon_j - \xi} S_j^- + \frac{I}{4} - \frac{\varepsilon_j^2 + \xi^2}{\varepsilon_j^2 - \xi^2}\frac{I}{2}. \end{aligned} \tag{4.9}$$

Define

$$\tau_j^{(1)} = \sum_{k\neq j}^{\mathcal{L}} \frac{4\varepsilon_j^2}{\varepsilon_j^2 - \varepsilon_k^2} S_j^z S_k^z + \sum_{k\neq j}^{\mathcal{L}} \frac{2\varepsilon_j^2}{\varepsilon_j^2 - \varepsilon_k^2}\left(\frac{\varepsilon_k + \xi}{\varepsilon_j + \xi} S_j^+ S_k^- + \frac{\varepsilon_k - \xi}{\varepsilon_j - \xi} S_j^- S_k^+\right) +$$
$$+ \frac{2\alpha\varepsilon_j^2}{\varepsilon_j^2 - \xi^2} S_j^z + \frac{\gamma\varepsilon_j^2}{\varepsilon_j + \xi} S_j^+ - \frac{\lambda\varepsilon_j^2}{\varepsilon_j - \xi} S_j^-.$$

We see that, up to a constant term, it is the same expression as (4.9). Consider the following local transformation on the lth space in the tensor product (2.5):

$$U_l = \mathrm{diag}\left(\sqrt{\frac{\varepsilon_l+\xi}{\varepsilon_l-\xi}}, 1\right).$$

Under these transformations we have

$$\begin{aligned} U_l S_l^z U_l^{-1} &= S_l^z, \\ U_l S_l^+ U_l^{-1} &= \sqrt{\frac{\varepsilon_l+\xi}{\varepsilon_l-\xi}} S_l^+, \\ U_l S_l^- U_l^{-1} &= \sqrt{\frac{\varepsilon_j-\xi}{\varepsilon_l+\xi}} S_l^-. \end{aligned}$$

Under the global transformation $U = U_1 U_2 \cdots U_{\mathcal{L}}$ we define

$$\begin{aligned} \tau_j^{(2)} &= U\tau_j^{(1)}U^{-1} = \\ &= \sum_{k\neq j}^{\mathcal{L}} \frac{4\varepsilon_j^2}{\varepsilon_j^2-\varepsilon_k^2} S_j^z S_k^z + \sum_{k\neq j}^{\mathcal{L}} \frac{2\varepsilon_j^2}{\varepsilon_j^2-\varepsilon_k^2} \frac{\sqrt{\varepsilon_k^2-\xi^2}}{\sqrt{\varepsilon_j^2-\xi^2}} (S_j^+S_k^- + S_j^-S_k^+) + \\ &\quad + \frac{2\alpha\varepsilon_j^2}{\varepsilon_j^2-\xi^2} S_j^z + \frac{\gamma\varepsilon_j^2}{\sqrt{\varepsilon_j^2-\xi^2}} S_j^+ - \frac{\lambda\varepsilon_j^2}{\sqrt{\varepsilon_j^2-\xi^2}} S_j^-. \end{aligned}$$

Next simply rescale to obtain

$$\begin{aligned} \tau_j^{(3)} &= \frac{\varepsilon_j^2-\xi^2}{\varepsilon_j^2}\tau_j^{(2)} = \\ &= \sum_{k\neq j}^{\mathcal{L}} \frac{4(\varepsilon_j^2-\xi^2)}{\varepsilon_j^2-\varepsilon_k^2} S_j^z S_k^z + \sum_{k\neq j}^{\mathcal{L}} \frac{2\sqrt{\varepsilon_j^2-\xi^2}\sqrt{\varepsilon_k^2-\xi^2}}{\varepsilon_j^2-\varepsilon_k^2} (S_j^+S_k^- + S_j^-S_k^+) + 2\alpha S_j^z + \\ &\quad + \gamma\sqrt{\varepsilon_j^2-\xi^2} S_j^+ - \lambda\sqrt{\varepsilon_j^2-\xi^2} S_j^-. \end{aligned}$$

Now we apply a change of variables $\varepsilon_j \mapsto \sqrt{\varepsilon_j^2 + \xi^2}$ to obtain

$$\begin{aligned}\tau_j^* &= \sum_{k\neq j}^{\mathcal{L}} \frac{4\varepsilon_j^2}{\varepsilon_j^2 - \varepsilon_k^2} S_j^z S_k^z + \sum_{k\neq j}^{\mathcal{L}} \frac{2\varepsilon_j\varepsilon_k}{\varepsilon_j^2 - \varepsilon_k^2}(S_j^+ S_k^- + S_j^- S_k^+) + 2\alpha S_j^z + \\ &+ \gamma\varepsilon_j S_j^+ - \lambda\varepsilon_j S_j^- = \\ &= \left.\frac{\varepsilon_j\tau_j}{(\varepsilon_j - \xi)(\varepsilon_j + \xi)}\right|_{\xi=0} + \frac{I}{4}.\end{aligned} \tag{4.10}$$

This affirms that we may set $\xi = 0$ without loss of generality.

We refer to the set of mutually commuting conserved operators $\{\tau_j^* : j = 1, \dots, \mathcal{L}\}$ as the open, rational Richardson–Gaudin system in the spin-1/2 case. Note that the coefficients of the $S_j^z S_k^z$ terms in (4.10) are not antisymmetric with respect to the interchange of indices j and k. This distinguishes this set of commuting operators from those obtained by the Gaudin algebra approach [RDO10, RBN14, CRBN15].

4.2.4 Hamiltonian

Let now construct the Hamiltonian from these conserved operators. Consider

$$\begin{aligned}\sum_{j=1}^{\mathcal{L}} \varepsilon_j^{-2}\tau_j^* &= \sum_{j=1}^{\mathcal{L}}\sum_{k\neq j}^{\mathcal{L}} \frac{4}{\varepsilon_j^2 - \varepsilon_k^2} S_j^z S_k^z + \sum_{j=1}^{\mathcal{L}}\sum_{k\neq j}^{\mathcal{L}} \frac{2\varepsilon_j^{-1}\varepsilon_k}{\varepsilon_j^2 - \varepsilon_k^2}\left(S_j^+ S_k^- + S_j^- S_k^+\right) + 2\alpha\sum_{j=1}^{\mathcal{L}} \varepsilon_j^{-2} S_j^z + \\ &+ \gamma\sum_{j=1}^{\mathcal{L}} \varepsilon_j^{-1} S_j^+ - \lambda\sum_{j=1}^{\mathcal{L}} \varepsilon_j^{-1} S_j^- = \\ &= 4\sum_{j=1}^{\mathcal{L}}\sum_{k=j+1}^{\mathcal{L}} \left(\frac{1}{\varepsilon_j^2 - \varepsilon_k^2} - \frac{1}{\varepsilon_j^2 - \varepsilon_k^2}\right) S_j^z S_k^z + \\ &+ 2\sum_{j=1}^{\mathcal{L}}\sum_{k=j+1}^{\mathcal{L}} \left(\frac{\varepsilon_j^{-1}\varepsilon_k}{\varepsilon_j^2 - \varepsilon_k^2} - \frac{\varepsilon_k^{-1}\varepsilon_j}{\varepsilon_j^2 - \varepsilon_k^2}\right)\left(S_j^+ S_k^- + S_j^- S_k^+\right) + \\ &+ 2\alpha\sum_{j=1}^{\mathcal{L}} \varepsilon_j^{-2} S_j^z + \gamma\sum_{j=1}^{\mathcal{L}} \varepsilon_j^{-1} S_j^+ - \lambda\sum_{j=1}^{\mathcal{L}} \varepsilon_j^{-1} S_j^- = \\ &= 2\alpha\sum_{j=1}^{\mathcal{L}} \varepsilon_j^{-2} S_j^z - 2\sum_{j=1}^{\mathcal{L}}\sum_{k=j+1}^{\mathcal{L}} \varepsilon_j^{-1}\varepsilon_k^{-1}\left(S_j^+ S_k^- + S_j^- S_k^+\right) + \\ &+ \gamma\sum_{j=1}^{\mathcal{L}} \varepsilon_j^{-1} S_j^+ - \lambda\sum_{j=1}^{\mathcal{L}} \varepsilon_j^{-1} S_j^-.\end{aligned}$$

Setting $\lambda = -\gamma$, and making the change of variable $z_j = \varepsilon_j^{-1}$ we obtain

$$\begin{aligned} H &= \frac{1}{2\alpha} \sum_{j=1}^{\mathcal{L}} \varepsilon_j^{-2} \tau_j^* = \\ &= \sum_{j=1}^{\mathcal{L}} z_j^2 S_j^z - \frac{1}{\alpha} \sum_{j=1}^{\mathcal{L}} \sum_{k=j+1}^{\mathcal{L}} z_j z_k \left(S_j^+ S_k^- + S_j^- S_k^+ \right) + \frac{\gamma}{2\alpha} \sum_{j=1}^{\mathcal{L}} z_j \left(S_j^+ + S_j^- \right). \end{aligned} \tag{4.11}$$

4.2.5 Physical interpretation

In this section we discuss a physical interpretation of the Hamiltonian (4.11) constructed above. It turns out that it is simply the $p + ip$ pairing Hamiltonian with extra terms of a specific form, which can be interpreted as interaction of the system with its environment.

Let us first introduce the isolated pairing mode not interacting with the environment. Let $c_{\mathbf{k}}, c_{\mathbf{k}}^\dagger$ denote the annihilation and creation operators for two-dimensional fermions of mass m and momentum $\mathbf{k} = (k_x, k_y)$. Then the $p + ip$ pairing Hamiltonian is

$$H_0 = \sum_{\mathbf{k}} \frac{|\mathbf{k}|^2}{2m} c_{\mathbf{k}}^\dagger c_{\mathbf{k}} - \frac{G}{4m} \sum_{\mathbf{k} \neq \pm \mathbf{k}'} (k_x + ik_y)(k'_x - ik'_y) c_{\mathbf{k}}^\dagger c_{-\mathbf{k}}^\dagger c_{-\mathbf{k}'} c_{\mathbf{k}'},$$

where $G \in \mathbb{R}$ is a dimensionless coupling constant and the summation is taken over all momentum states $\mathbf{k}$. The annihilation and creation operators $c_{\mathbf{k}}, c_{\mathbf{k}}^\dagger$ satisfy the canonical anticommutation relations

$$\{c_{\mathbf{k}}, c_{\mathbf{k}'}\} = \{c_{\mathbf{k}}^\dagger, c_{\mathbf{k}'}^\dagger\} = 0, \ \{c_{\mathbf{k}}, c_{\mathbf{k}'}^\dagger\} = \delta_{\mathbf{k}\mathbf{k}'} I.$$

Now consider a more general Hamiltonian with extra terms

$$H = H_0 + \frac{\Gamma}{2} \sum_{\mathbf{k}} \Big((k_x + ik_y) c_{\mathbf{k}}^\dagger c_{-\mathbf{k}}^\dagger + (k_x - ik_y) c_{-\mathbf{k}} c_{\mathbf{k}} \Big), \tag{4.12}$$

where $\Gamma \in \mathbb{R}$ is a constant. We note that this Hamiltonian is Hermitian, and the extra terms can be interpreted as creation and annihilation of pairs of fermions, resulting from interaction with the environment. This interaction is not general but rather has a specific momentum-dependent coupling similar to that occurring in H_0. It is important to distinguish this type of interaction with the environment from other examples, e.g. [BM78] in the context of a heat bath, which facilitate a notion of entanglement with the environment. In our model there is no entanglement between the system and the environment, because the state space for the environment is not explicitly defined.

We now restrict to the Hilbert subspace that allows only paired particle states. By imposing this restriction, we do not consider states on which the operators in the interaction term in the Hamiltonian (i.e. the second term) has trivial action. On this subspace the following equality is satisfied:

$$2c^\dagger_{\mathbf{k}}c_{\mathbf{k}}c^\dagger_{-\mathbf{k}}c_{-\mathbf{k}} = c^\dagger_{\mathbf{k}}c_{\mathbf{k}} + c^\dagger_{-\mathbf{k}}c_{-\mathbf{k}}. \tag{4.13}$$

Set $z_{\mathbf{k}} = |\mathbf{k}|$ and $k_x + ik_y = |\mathbf{k}|\exp(i\phi_{\mathbf{k}})$. Introduce the following notation:

$$S^+_{\mathbf{k}} = \exp(i\phi_{\mathbf{k}})c^\dagger_{\mathbf{k}}c^\dagger_{-\mathbf{k}},\ S^-_{\mathbf{k}} = \exp(-i\phi_{\mathbf{k}})c_{-\mathbf{k}}c_{\mathbf{k}},\ S^z_{\mathbf{k}} = c^\dagger_{\mathbf{k}}c^\dagger_{-\mathbf{k}}c_{-\mathbf{k}}c_{\mathbf{k}} - \frac{I}{2}.$$

Remark 4.6. *On this restricted subspace, one may verify the* $\mathfrak{su}(2)$ *algebra commutation relations*

$$[S^z_{\mathbf{k}}, S^\pm_{\mathbf{k}}] = \pm S^\pm_{\mathbf{k}},\ [S^+_{\mathbf{k}}, S^-_{\mathbf{k}}] = 2S^z_{\mathbf{k}}.$$

We now use integers $k = 1, \ldots, \mathcal{L}$ to enumerate the unblocked pairs of momentum states ($\mathbf{k}$ and $-\mathbf{k}$). Working in units such that $m = 1$, using equation (4.13) and ignoring the constant term $\frac{1}{2}\sum_{k=1}^{\mathcal{L}} z_k^2$, we obtain

$$H_0 = \sum_{k=1}^{\mathcal{L}} z_k^2 S^z_k - G\sum_{j=1}^{\mathcal{L}}\sum_{k\neq j}^{\mathcal{L}} z_k z_j S^+_k S^-_j,$$

which exhibits $\mathfrak{u}(1)$-symmetry associated with the operator $S^z = \sum_{k=1}^{\mathcal{L}} S^z_k$. The full Hamiltonian (4.12) can be therefore rewritten as

$$H = H_0 + \Gamma\sum_{k=1}^{\mathcal{L}} z_k\left(S^+_k + S^-_k\right). \tag{4.14}$$

We see that (4.11) is equivalent to (4.14) by identifying $\alpha = G^{-1}$ and $\gamma = 2\Gamma G^{-1}$. Thus, we have shown that the Hamiltonian (4.14) is integrable by means of the BQISM.

The Hamiltonian (4.14) no longer possesses $\mathfrak{u}(1)$-symmetry. Thus, the algebraic Bethe Ansatz can no longer be applied, due to the absence of an obvious reference state. In the following we apply the recently developed *off-diagonal Bethe Ansatz* [WYCS15] to derive the formulae for the eigenvalues of the conserved operators (4.10), the corresponding BAE and the energy spectrum (the eigenvalues of the Hamiltonian (4.14)).

4.3 Eigenvalues, Bethe Ansatz Equations and the energy spectrum

4.3.1 Eigenvalues

Let us rewrite the K-matrices (4.1) in the following form (using the notation from [HCYY15]):

$$K^-(u) = \xi^- + (u-\eta/2)\begin{pmatrix} 1 & \psi^- \\ \phi^- & -1 \end{pmatrix} = \xi^- + (u-\eta/2)\left(\psi^- S^+ + \phi^- S^- + 2S^z\right) =$$
$$= \xi^- + (u-\eta/2)\left(\frac{\psi^- + \phi^-}{2}\sigma^x + \frac{i(\psi^- - \phi^-)}{2}\sigma^y + \sigma^z\right),$$

where $\vec{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$ are the Pauli matrices. Analogously for $K^+(u)$. Thus,

$$K^-(u) = \xi^- + (u-\eta/2)\vec{h}_1 \cdot \vec{\sigma} \quad \text{with} \quad \vec{h}_1 = \left(\frac{\psi^- + \phi^-}{2}, \frac{i(\psi^- - \phi^-)}{2}, 1\right),$$
$$K^+(u) = \xi^+ + (u+\eta/2)\vec{h}_2 \cdot \vec{\sigma} \quad \text{with} \quad \vec{h}_2 = \left(\frac{\psi^+ + \phi^+}{2}, \frac{i(\psi^+ - \phi^+)}{2}, 1\right).$$

To match the notation from [WYCS15] we need to normalize the vectors $\vec{h}_1$ and $\vec{h}_2$. Compute the norms

$$|\vec{h}_1| = \sqrt{\frac{(\psi^- + \phi^-)^2}{4} - \frac{(\psi^- - \phi^-)^2}{4} + 1} = \sqrt{\psi^-\phi^- + 1},$$
$$|\vec{h}_2| = \sqrt{\psi^+\phi^+ + 1}.$$

Introduce the normalised vectors

$$\vec{h}_1^0 = \frac{\vec{h}_1}{\sqrt{\psi^-\phi^- + 1}}, \quad \vec{h}_2^0 = \frac{\vec{h}_2}{\sqrt{\psi^+\phi^+ + 1}}.$$

Now, the K-matrices can be written as

$$\begin{aligned}
K^-(u) &= \sqrt{\psi^-\phi^-+1}\left(\frac{\xi^-}{\sqrt{\psi^-\phi^-+1}} + (u-\eta/2)\vec{h}_1^0\cdot\vec{\sigma}\right),\\
K^+(u) &= \sqrt{\psi^+\phi^++1}\left(\frac{\xi^+}{\sqrt{\psi^+\phi^++1}} + (u+\eta/2)\vec{h}_2^0\cdot\vec{\sigma}\right).
\end{aligned}$$

Let $\{v_k \mid k = 1, 2, \ldots, \mathcal{L}\}$ denote a set of parameters that will be utilised to determine the eigenvalues of the transfer matrix (4.2). These are analogous to the parameters in (2.13) and (2.46), but in the off-diagonal case there is no obvious reference state. From [HCYY15], the formula for the eigenvalues of (4.2) is given by

$$\Lambda(u) = \sqrt{\psi^-\phi^-+1}\sqrt{\psi^+\phi^++1}\left[a(u)\frac{Q(u+\eta)}{Q(u)} + d(u)\frac{Q(u-\eta)}{Q(u)} + c(u-\eta/2)(u+\eta/2)\frac{F(u)}{Q(u)}\right],$$

where

$$\begin{aligned}
Q(u) &= \prod_{i=1}^{\mathcal{L}}(u-v_i)(u+v_i),\\
a(u) &= \frac{2u-\eta}{2u}\left(u+\frac{\xi^-}{\sqrt{\psi^-\phi^-+1}}+\frac{\eta}{2}\right)\left(u+\frac{\xi^+}{\sqrt{\psi^+\phi^++1}}+\frac{\eta}{2}\right)\times\\
&\quad\times\prod_{l=1}^{\mathcal{L}}\frac{(u-\varepsilon_l-\eta/2)(u+\varepsilon_l-\eta/2)}{(u-\varepsilon_l)(u+\varepsilon_l)},\\
d(u) &= \frac{2u+\eta}{2u}\left(u-\frac{\xi^-}{\sqrt{\psi^-\phi^-+1}}-\frac{\eta}{2}\right)\left(u-\frac{\xi^+}{\sqrt{\psi^+\phi^++1}}-\frac{\eta}{2}\right)\times\\
&\quad\times\prod_{l=1}^{\mathcal{L}}\frac{(u-\varepsilon_l+\eta/2)(u+\varepsilon_l+\eta/2)}{(u-\varepsilon_l)(u+\varepsilon_l)},\\
F(u) &= \prod_{i=1}^{\mathcal{L}}\frac{(u+\varepsilon_i-\eta/2)(u-\varepsilon_i-\eta/2)(u+\varepsilon_i+\eta/2)(u-\varepsilon_i+\eta/2)}{(u-\varepsilon_i)(u+\varepsilon_i)},\\
c &= 2\left(\vec{h}_1^0\cdot\vec{h}_2^0-1\right).
\end{aligned}$$

The constant c can be computed as

$$c = 2\left(\frac{\vec{h}_1\cdot\vec{h}_2}{\sqrt{\psi^-\phi^-+1}\sqrt{\psi^+\phi^++1}}-1\right) = 2\left(\frac{\frac{1}{2}(\psi^-\phi^++\phi^-\psi^+)+1}{\sqrt{\psi^-\phi^-+1}\sqrt{\psi^+\phi^++1}}-1\right).$$

Finally, we obtain

$$\begin{aligned}\Lambda(u) &= \sqrt{\psi^-\phi^-+1}\sqrt{\psi^+\phi^++1}\,\times\\ &\times\left[a(u)\prod_{i=1}^{\mathcal{L}}\frac{(u-v_i+\eta)(u+v_i+\eta)}{(u-v_i)(u+v_i)}+d(u)\prod_{i=1}^{\mathcal{L}}\frac{(u-v_i-\eta)(u+v_i-\eta)}{(u-v_i)(u+v_i)}+\right.\\ &\left.+\,c(u^2-\eta^2/4)\prod_{i=1}^{\mathcal{L}}\frac{\big((u+\varepsilon_i)^2-\eta^2/4\big)\big((u-\varepsilon_i)^2-\eta^2/4\big)}{(u^2-\varepsilon_i^2)(u^2-v_i^2)}\right],\end{aligned}\tag{4.15}$$

where

$$\begin{aligned}a(u) &= \frac{2u-\eta}{2u}\left(u+\frac{\xi^-}{\sqrt{\psi^-\phi^-+1}}+\frac{\eta}{2}\right)\left(u+\frac{\xi^+}{\sqrt{\psi^+\phi^++1}}+\frac{\eta}{2}\right)\times\\ &\quad\times\prod_{l=1}^{\mathcal{L}}\frac{(u-\varepsilon_l-\eta/2)(u+\varepsilon_l-\eta/2)}{(u-\varepsilon_l)(u+\varepsilon_l)},\\ d(u) &= \frac{2u+\eta}{2u}\left(u-\frac{\xi^-}{\sqrt{\psi^-\phi^-+1}}-\frac{\eta}{2}\right)\left(u-\frac{\xi^+}{\sqrt{\psi^+\phi^++1}}-\frac{\eta}{2}\right)\times\\ &\quad\times\prod_{l=1}^{\mathcal{L}}\frac{(u-\varepsilon_l+\eta/2)(u+\varepsilon_l+\eta/2)}{(u-\varepsilon_l)(u+\varepsilon_l)}.\end{aligned}$$

Remark 4.7. *Keeping in mind the parameters defined in (4.4), as a consistency check consider*

$$\begin{aligned}\Lambda(u)|_{\eta=0} &= (\psi\phi+1)\left(u-\frac{\xi}{\sqrt{\psi\phi+1}}\right)\left(u+\frac{\xi}{\sqrt{\psi\phi+1}}\right)+\\ &\quad+(\psi\phi+1)\left(u+\frac{\xi}{\sqrt{\psi\phi+1}}\right)\left(u-\frac{\xi}{\sqrt{\psi\phi+1}}\right)+\\ &\quad+(2\psi\phi+2-2(\psi\phi+1))\,u^2\prod_{i=1}^{\mathcal{L}}\frac{u^2-\varepsilon_i^2}{u^2-v_i^2}=\\ &=2\left((\psi\phi+1)u^2-\xi^2\right).\end{aligned}$$

Now consider $t(u)|_{\eta=0}$*:*

$$K^+(u)\big|_{\eta=0}=\begin{pmatrix}\xi+u & \psi u\\ \phi u & \xi-u\end{pmatrix},\quad K^-(u)\big|_{\eta=0}=\begin{pmatrix}-\xi+u & \psi u\\ \phi u & -\xi-u\end{pmatrix},\quad L_{al}(u)|_{\eta=0}=I.$$

Thus,

$$t(u)|_{\eta=0}=\mathrm{tr}_a\left(\begin{pmatrix}\xi+u & \psi u\\ \phi u & \xi-u\end{pmatrix}\begin{pmatrix}-\xi+u & \psi u\\ \phi u & -\xi-u\end{pmatrix}\right)=2\left((\psi\phi+1)u^2-\xi^2\right)I.$$

That is, the constant terms in the eigenvalue and the transfer matrix coincide.

4.3.2 Quasi-classical limit of the eigenvalues

The eigenvalues in the quasi-classical limit ($\eta \to 0$) are constructed as usual from (4.15):

$$\lim_{u\to\varepsilon_j}(u-\varepsilon_j)\Lambda(u) = \eta^2\lambda_j + \mathcal{O}(\eta^3).$$

We compute this limit assuming the same dependence (4.4) as for the conserved operators:

$$\begin{aligned}
&\lim_{u\to\varepsilon_j}(u-\varepsilon_j)\prod_{l=1}^{\mathcal{L}}\frac{(u+\varepsilon_l-\eta/2)(u-\varepsilon_l-\eta/2)}{(u-\varepsilon_l)(u+\varepsilon_l)} = \\
&\qquad = -\frac{\eta}{2}+\frac{\eta^2}{4}\left(\frac{1}{2\varepsilon_j}+\sum_{k\neq j}^{\mathcal{L}}\left(\frac{1}{\varepsilon_j-\varepsilon_k}+\frac{1}{\varepsilon_j+\varepsilon_k}\right)\right)+\mathcal{O}(\eta^3),\\
&\lim_{u\to\varepsilon_j}(u-\varepsilon_j)\prod_{l=1}^{\mathcal{L}}\frac{(u+\varepsilon_l+\eta/2)(u-\varepsilon_l+\eta/2)}{(u-\varepsilon_l)(u+\varepsilon_l)} = \\
&\qquad = \frac{\eta}{2}+\frac{\eta^2}{4}\left(\frac{1}{2\varepsilon_j}+\sum_{k\neq j}^{\mathcal{L}}\left(\frac{1}{\varepsilon_j-\varepsilon_k}+\frac{1}{\varepsilon_j+\varepsilon_k}\right)\right)+\mathcal{O}(\eta^3),\\
&\sqrt{\psi^-\phi^-+1} = \sqrt{\psi\phi+1}\left(1+\frac{\eta}{2}\frac{\mu\psi+\delta\phi}{\psi\phi+1}\right)+\mathcal{O}(\eta^2),\\
&\sqrt{\psi^+\phi^+ +1} = \sqrt{\psi\phi+1}\left(1+\frac{\eta}{2}\frac{\lambda\psi+\gamma\phi}{\psi\phi+1}\right)+\mathcal{O}(\eta^2).
\end{aligned}$$

Consider the expansion of the first two terms in (4.15) up to second order in η:

$$\begin{aligned}
&\lim_{u\to\varepsilon_j}(u-\varepsilon_j)\sqrt{\psi^-\phi^-+1}\sqrt{\psi^+\phi^++1}\,a(u) = \\
&\quad = -\frac{\eta}{2}\left(\varepsilon_j^2(\psi\phi+1)-\xi^2\right) - \\
&\qquad -\frac{\eta^2}{2}\left(\frac{\xi^2}{2\varepsilon_j}+\frac{\varepsilon_j}{2}(\psi\phi+1)+\frac{\varepsilon_j^2}{2}\big((\lambda+\mu)\psi+(\gamma+\delta)\phi\big)+(\alpha+\beta)\varepsilon_j\sqrt{\psi\phi+1}-\right.\\
&\qquad\qquad \left.-\frac{\xi\varepsilon_j}{2\sqrt{\psi\phi+1}}\big((\lambda-\mu)\psi+(\gamma-\delta)\phi\big)-\xi(\alpha-\beta)\right)+\\
&\qquad +\frac{\eta^2}{4}\left(\varepsilon_j^2(\psi\phi+1)-\xi^2\right)\left(\frac{1}{2\varepsilon_j}+\sum_{k\neq j}^{\mathcal{L}}\left(\frac{1}{\varepsilon_j-\varepsilon_k}+\frac{1}{\varepsilon_j+\varepsilon_k}\right)\right)+\mathcal{O}(\eta^3),\\
&\lim_{u\to\varepsilon_j}(u-\varepsilon_j)\sqrt{\psi^-\phi^-+1}\sqrt{\psi^+\phi^++1}\,d(u) =
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\eta}{2}\left(\varepsilon_j^2(\psi\phi+1)-\xi^2\right)+\\
&+\frac{\eta^2}{2}\Bigg(-\frac{\xi^2}{2\varepsilon_j}-\frac{\varepsilon_j}{2}(\psi\phi+1)+\frac{\varepsilon_j^2}{2}\big((\lambda+\mu)\psi+(\gamma+\delta)\phi\big)-(\alpha+\beta)\varepsilon_j\sqrt{\psi\phi+1}+\\
&\qquad+\frac{\xi\varepsilon_j}{2\sqrt{\psi\phi+1}}\big((\lambda-\mu)\psi+(\gamma-\delta)\phi\big)-\xi(\alpha-\beta)\Bigg)+\\
&+\frac{\eta^2}{4}\left(\varepsilon_j^2(\psi\phi+1)-\xi^2\right)\left(\frac{1}{2\varepsilon_j}+\sum_{k\neq j}^{\mathcal{L}}\left(\frac{1}{\varepsilon_j-\varepsilon_k}+\frac{1}{\varepsilon_j+\varepsilon_k}\right)\right)+\mathcal{O}(\eta^3).
\end{aligned}
$$

Also

$$
\begin{aligned}
\prod_{i=1}^{\mathcal{L}}\frac{(u-v_i+\eta)(u+v_i+\eta)}{(u-v_i)(u+v_i)}&=1+\eta\sum_{i=1}^{\mathcal{L}}\left(\frac{1}{u-v_i}+\frac{1}{u+v_i}\right)+\mathcal{O}(\eta^2),\\
\prod_{i=1}^{\mathcal{L}}\frac{(u-v_i-\eta)(u+v_i-\eta)}{(u-v_i)(u+v_i)}&=1-\eta\sum_{i=1}^{\mathcal{L}}\left(\frac{1}{u-v_i}+\frac{1}{u+v_i}\right)+\mathcal{O}(\eta^2).
\end{aligned}
$$

Combining these calculations then leads to

$$
\begin{aligned}
&\lim_{u\to\varepsilon_j}(u-\varepsilon_j)\sqrt{\psi^-\phi^-+1}\sqrt{\psi^+\phi^++1}\,a(u)\prod_{i=1}^{\mathcal{L}}\frac{(u-v_i+\eta)(u+v_i+\eta)}{(u-v_i)(u+v_i)}=\\
&\quad=-\frac{\eta}{2}\big(\varepsilon_j^2(\psi\phi+1)-\xi^2\big)+\\
&\quad+\frac{\eta^2}{2}\Bigg[-\big(\varepsilon_j^2(\psi\phi+1)-\xi^2\big)\sum_{i=1}^{\mathcal{L}}\frac{2\varepsilon_j}{\varepsilon_j^2-v_i^2}+\frac{1}{2}\big(\varepsilon_j^2(\psi\phi+1)-\xi^2\big)\sum_{k\neq j}^{\mathcal{L}}\frac{2\varepsilon_j}{\varepsilon_j^2-\varepsilon_k^2}-\\
&\qquad-\frac{\varepsilon_j}{4}(\psi\phi+1)-\frac{3\xi^2}{4\varepsilon_j}-\frac{\varepsilon_j^2}{2}\big((\lambda+\mu)\psi+(\gamma+\delta)\phi\big)-(\alpha+\beta)\varepsilon_j\sqrt{\psi\phi+1}+\\
&\qquad+\frac{\xi\varepsilon_j}{2\sqrt{\psi\phi+1}}\big((\lambda-\mu)\psi+(\gamma-\delta)\phi\big)+\xi(\alpha-\beta)\Bigg]+\mathcal{O}(\eta^3),\\
&\lim_{u\to\varepsilon_j}(u-\varepsilon_j)\sqrt{\psi^-\phi^-+1}\sqrt{\psi^+\phi^++1}\,d(u)\prod_{i=1}^{\mathcal{L}}\frac{(u-v_i-\eta)(u+v_i-\eta)}{(u-v_i)(u+v_i)}=\\
&\quad=\frac{\eta}{2}\big(\varepsilon_j^2(\psi\phi+1)-\xi^2\big)+\\
&\quad+\frac{\eta^2}{2}\Bigg[-\big(\varepsilon_j^2(\psi\phi+1)-\xi^2\big)\sum_{i=1}^{\mathcal{L}}\frac{2\varepsilon_j}{\varepsilon_j^2-v_i^2}+\frac{1}{2}\big(\varepsilon_j^2(\psi\phi+1)-\xi^2\big)\sum_{k\neq j}^{\mathcal{L}}\frac{2\varepsilon_j}{\varepsilon_j^2-\varepsilon_k^2}-\\
&\qquad-\frac{\varepsilon_j}{4}(\psi\phi+1)-\frac{3\xi^2}{4\varepsilon_j}+\frac{\varepsilon_j^2}{2}\big((\lambda+\mu)\psi+(\gamma+\delta)\phi\big)-(\alpha+\beta)\varepsilon_j\sqrt{\psi\phi+1}+\\
&\qquad+\frac{\xi\varepsilon_j}{2\sqrt{\psi\phi+1}}\big((\lambda-\mu)\psi+(\gamma-\delta)\phi\big)-\xi(\alpha-\beta)\Bigg]+\mathcal{O}(\eta^3).
\end{aligned}
$$

Finally, the sum of the first two terms in (4.15) can be expressed as

$$\lim_{u\to\varepsilon_j}(u-\varepsilon_j)\sqrt{\psi^-\phi^-+1}\sqrt{\psi^+\phi^++1}\times$$
$$\times\left[a(u)\prod_{i=1}^{\mathcal{L}}\frac{(u-v_i+\eta)(u+v_i+\eta)}{(u-v_i)(u+v_i)}+d(u)\prod_{i=1}^{\mathcal{L}}\frac{(u-v_i-\eta)(u+v_i-\eta)}{(u-v_i)(u+v_i)}\right]=$$
$$=\eta^2\Bigg[-\big(\varepsilon_j^2(\psi\phi+1)-\xi^2\big)\sum_{i=1}^{\mathcal{L}}\left(\frac{1}{\varepsilon_j-v_i}+\frac{1}{\varepsilon_j+v_i}\right)+$$
$$+\frac{1}{2}\big(\varepsilon_j^2(\psi\phi+1)-\xi^2\big)\sum_{k\neq j}^{\mathcal{L}}\left(\frac{1}{\varepsilon_j-\varepsilon_k}+\frac{1}{\varepsilon_j+\varepsilon_k}\right)-\frac{\varepsilon_j}{4}(\psi\phi+1)-$$
$$-\frac{3\xi^2}{4\varepsilon_j}-(\alpha+\beta)\varepsilon_j\sqrt{\psi\phi+1}+\frac{\xi\varepsilon_j}{2\sqrt{\psi\phi+1}}\big((\lambda-\mu)\psi+(\gamma-\delta)\phi\big)\Bigg]=$$
$$=\eta^2\Bigg[-\big(\varepsilon_j^2(\psi\phi+1)-\xi^2\big)\sum_{i=1}^{\mathcal{L}}\frac{2\varepsilon_j}{\varepsilon_j^2-v_i^2}+\big(\varepsilon_j^2(\psi\phi+1)-\xi^2\big)\sum_{k\neq j}^{\mathcal{L}}\frac{\varepsilon_j}{\varepsilon_j^2-\varepsilon_k^2}+$$
$$+\frac{3\varepsilon_j}{4}(\psi\phi+1)-\frac{3\xi^2}{4\varepsilon_j}-\varepsilon_j(\psi\phi+1)-(\alpha+\beta)\varepsilon_j\sqrt{\psi\phi+1}+$$
$$+\frac{\xi\varepsilon_j}{2\sqrt{\psi\phi+1}}\big((\lambda-\mu)\psi+(\gamma-\delta)\phi\big)\Bigg]=$$
$$=\eta^2\Bigg[\frac{\big(\varepsilon_j^2(\psi\phi+1)-\xi^2\big)}{\varepsilon_j}\left(\sum_{k\neq j}^{\mathcal{L}}\frac{\varepsilon_j^2}{\varepsilon_j^2-\varepsilon_k^2}-\sum_{i=1}^{\mathcal{L}}\frac{2\varepsilon_j^2}{\varepsilon_j^2-v_i^2}+\frac{3}{4}\right)-\varepsilon_j(\psi\phi+1)-$$
$$-(\alpha+\beta)\varepsilon_j\sqrt{\psi\phi+1}+\frac{\xi\varepsilon_j}{2\sqrt{\psi\phi+1}}\big((\lambda-\mu)\psi+(\gamma-\delta)\phi\big)\Bigg]+\mathcal{O}(\eta^3).$$

The third term of (4.15), reproduced here for convenience,

$$\lim_{u\to\varepsilon_j}(u-\varepsilon_j)\Bigg[\sqrt{\psi^-\phi^-+1}\sqrt{\psi^+\phi^++1}\,c(u^2-\eta^2/4)\times$$
$$\times\prod_{i=1}^{\mathcal{L}}\frac{\big((u+\varepsilon_i)^2-\eta^2/4\big)\big((u-\varepsilon_i)^2-\eta^2/4\big)}{(u^2-\varepsilon_i^2)(u^2-v_i^2)}\Bigg],$$

is computed as follows. First, expand the product in powers of η:

$$\prod_{i=1}^{\mathcal{L}}\frac{\big((u+\varepsilon_i)^2-\eta^2/4\big)\big((u-\varepsilon_i)^2-\eta^2/4\big)}{(u^2-\varepsilon_i^2)(u^2-v_i^2)}=$$
$$=\prod_{i=1}^{\mathcal{L}}\frac{(u+\varepsilon_i)^2(u-\varepsilon_i)^2-(\eta^2/4)\big((u+\varepsilon_i)^2+(u-\varepsilon_i)^2\big)+\mathcal{O}(\eta^3)}{(u+\varepsilon_i)(u+\varepsilon_i)(u^2-v_i^2)}=$$

$$= \prod_{i=1}^{\mathcal{L}} \frac{u^2 - \varepsilon_i^2}{u^2 - v_i^2} - \frac{\eta^2}{4} \sum_{i,k:i\neq k}^{\mathcal{L}} \frac{u^2 - \varepsilon_i^2}{u^2 - v_i^2} \frac{1}{u^2 - v_k^2} \left(\frac{u + \varepsilon_k}{u - \varepsilon_k} + \frac{u - \varepsilon_k}{u + \varepsilon_k} \right) + \mathcal{O}(\eta^3).$$

Thus,

$$\lim_{u \to \varepsilon_j} (u - \varepsilon_j) \prod_{i=1}^{\mathcal{L}} \frac{((u + \varepsilon_i)^2 - \eta^2/4)((u - \varepsilon_i)^2 - \eta^2/4)}{(u^2 - \varepsilon_i^2)(u^2 - v_i^2)} =$$
$$= -\frac{\eta^2}{4} \sum_{i\neq j}^{\mathcal{L}} \frac{\varepsilon_j^2 - \varepsilon_i^2}{\varepsilon_j^2 - v_i^2} \frac{2\varepsilon_j}{\varepsilon_j^2 - v_j^2} + \mathcal{O}(\eta^3).$$

This term already gives the multiple of η^2, so we just need to consider the constant contribution from the other multiples. Consider

$$\sqrt{\psi^- \phi^- + 1}\sqrt{\psi^+ \phi^+ + 1}\, c \Big|_{\eta=0} =$$
$$= \left((\psi^- \phi^+ + \phi^- \psi^+) + 2 - 2\sqrt{\psi^- \phi^- + 1}\sqrt{\psi^+ \phi^+ + 1} \right)\Big|_{\eta=0} =$$
$$= 2\psi\phi + 2 - 2(\psi\phi + 1) = 0.$$

Thus, there will be no contribution in the eigenvalues from the third term in (4.15). Finally, we obtain the eigenvalues of the conserved operators (4.8) as

$$\begin{aligned} \lambda_j = {} & \frac{(\varepsilon_j^2(\psi\phi + 1) - \xi^2)}{\varepsilon_j} \left(\sum_{k\neq j}^{\mathcal{L}} \frac{\varepsilon_j^2}{\varepsilon_j^2 - \varepsilon_k^2} - \sum_{i=1}^{\mathcal{L}} \frac{2\varepsilon_j^2}{\varepsilon_j^2 - v_i^2} + \frac{3}{4} \right) - \varepsilon_j(\psi\phi + 1) - \\ & - (\alpha + \beta)\varepsilon_j \sqrt{\psi\phi + 1} + \frac{\xi\varepsilon_j}{2\sqrt{\psi\phi + 1}}((\lambda - \mu)\psi + (\gamma - \delta)\phi). \end{aligned} \tag{4.16}$$

Remark 4.8. *In the case when $K^-(u)$ is diagonal, by setting $\beta = \psi = \phi = \delta = \mu = 0$ we obtain*

$$\lambda_j = \frac{\varepsilon_j^2 - \xi^2}{\varepsilon_j} \left(\sum_{k\neq j}^{\mathcal{L}} \frac{\varepsilon_j^2}{\varepsilon_j^2 - \varepsilon_k^2} - \sum_{i=1}^{\mathcal{L}} \frac{2\varepsilon_j^2}{\varepsilon_j^2 - v_i^2} + \frac{3}{4} - \frac{(\alpha + 1)\varepsilon_j^2}{\varepsilon_j^2 - \xi^2} \right),$$

$$\frac{\varepsilon_j \lambda_j}{\varepsilon_j^2 - \xi^2} = \sum_{k\neq j}^{\mathcal{L}} \frac{\varepsilon_j^2}{\varepsilon_j^2 - \varepsilon_k^2} - \sum_{i=1}^{\mathcal{L}} \frac{2\varepsilon_j^2}{\varepsilon_j^2 - v_i^2} + \frac{3}{4} - \frac{(\alpha + 1)\varepsilon_j^2}{\varepsilon_j^2 - \xi^2}.$$

As for the conserved operators, we can set $\xi = 0$ by a suitable variable change

$$\frac{\varepsilon_j \lambda_j}{\varepsilon_j^2 - \xi^2}\Big|_{\xi=0} = \sum_{k\neq j}^{\mathcal{L}} \frac{\varepsilon_j^2}{\varepsilon_j^2 - \varepsilon_k^2} - \sum_{i=1}^{\mathcal{L}} \frac{2\varepsilon_j^2}{\varepsilon_j^2 - v_i^2} - \frac{1}{4} - \alpha.$$

Thus, the eigenvalues of τ_j^* (4.10) can be found as

$$\lambda_j^* = \frac{\varepsilon_j \lambda_j}{\varepsilon_j^2 - \xi^2}\bigg|_{\xi=0} + \frac{1}{4} = \sum_{k\neq j}^{\mathcal{L}} \frac{\varepsilon_j^2}{\varepsilon_j^2 - \varepsilon_k^2} - \sum_{i=1}^{\mathcal{L}} \frac{2\varepsilon_j^2}{\varepsilon_j^2 - v_i^2} - \alpha. \tag{4.17}$$

4.3.3 Bethe Ansatz Equations

The eigenvalue expression for $\Lambda(u)$ given in (4.15) is undefined for $u = v_k$, for each $k = 1, 2, \ldots, \mathcal{L}$. Assuming that the v_k are all distinct, analyticity of $\Lambda(u)$ requires that $\lim\limits_{u\to v_k} \Lambda(u)$ must be finite for each $k = 1, 2, \ldots, \mathcal{L}$. This requirement equates to evaluating the residue of $\Lambda(u)$ at $u = v_k$, and the resulting constraints on the v_k are referred to as the BAE, as in the diagonal case. The BAE are equivalent to

$$\lim_{u\to v_k} (u - v_k)\Lambda(u) = 0. \tag{4.18}$$

Compute (4.18) from (4.15):

$$\begin{aligned}
\sqrt{\psi^-\phi^- + 1}\sqrt{\psi^+\phi^+ + 1}\Bigg[a(v_k)\frac{\eta(2v_k+\eta)}{2v_k}\prod_{i\neq k}^{\mathcal{L}} \frac{(v_k - v_i + \eta)(v_k + v_i + \eta)}{(v_k - v_i)(v_k + v_i)} + \\
+ d(v_k)\frac{-\eta(2v_k-\eta)}{2v_k}\prod_{i\neq k}^{\mathcal{L}} \frac{(v_k - v_i - \eta)(v_k + v_i - \eta)}{(v_k - v_i)(v_k + v_i)} + \\
+ c(v_k^2 - \eta^2/4)\frac{((v_k+\varepsilon_k)^2 - \eta^2/4)((v_k-\varepsilon_k)^2 - \eta^2/4)}{(v_k^2 - \varepsilon_k^2)2v_k} \times \\
\times \prod_{i\neq k}^{\mathcal{L}} \frac{((v_k+\varepsilon_i)^2 - \eta^2/4)((v_k-\varepsilon_i)^2 - \eta^2/4)}{(v_k^2 - \varepsilon_i^2)(v_k^2 - v_i^2)}\Bigg] = 0.
\end{aligned}$$

Substituting the expressions for $a(u)$, $d(u)$ and c:

$$\begin{aligned}
&\frac{2v_k - \eta}{2v_k}\left(v_k\sqrt{\psi^-\phi^- + 1} + \xi^- + \frac{\eta}{2}\sqrt{\psi^-\phi^- + 1}\right)\left(v_k\sqrt{\psi^+\phi^+ + 1} + \xi^+ + \frac{\eta}{2}\sqrt{\psi^+\phi^+ + 1}\right) \times \\
&\quad\times \prod_{l=1}^{\mathcal{L}} \frac{(v_k - \varepsilon_l - \eta/2)(v_k + \varepsilon_l - \eta/2)}{(v_k - \varepsilon_l)(v_k + \varepsilon_l)} \frac{\eta(2v_k+\eta)}{2v_k} \prod_{i\neq k}^{\mathcal{L}} \frac{(v_k - v_i + \eta)(v_k + v_i + \eta)}{(v_k - v_i)(v_k + v_i)} + \\
&+ \frac{2v_k + \eta}{2v_k}\left(v_k\sqrt{\psi^-\phi^- + 1} - \xi^- - \frac{\eta}{2}\sqrt{\psi^-\phi^- + 1}\right) \times \\
&\quad\times \left(v_k\sqrt{\psi^+\phi^+ + 1} - \xi^+ - \frac{\eta}{2}\sqrt{\psi^+\phi^+ + 1}\right) \times
\end{aligned}$$

$$
\begin{aligned}
&\times \prod_{l=1}^{\mathcal{L}} \frac{(v_k-\varepsilon_l+\eta/2)(v_k+\varepsilon_l+\eta/2)}{(v_k-\varepsilon_l)(v_k+\varepsilon_l)} \frac{-\eta(2v_k-\eta)}{2v_k} \prod_{i\neq k}^{\mathcal{L}} \frac{(v_k-v_i-\eta)(v_k+v_i-\eta)}{(v_k-v_i)(v_k+v_i)} + \\
&+\left((\psi^-\phi^+ + \phi^-\psi^+) + 2 - 2\sqrt{\psi^-\phi^- + 1}\sqrt{\psi^+\phi^+ + 1}\right)(v_k^2-\eta^2/4)\times \\
&\times \frac{((v_k+\varepsilon_k)^2-\eta^2/4)((v_k-\varepsilon_k)^2-\eta^2/4)}{(v_k^2-\varepsilon_k^2)2v_k} \prod_{i\neq k}^{\mathcal{L}} \frac{((v_k+\varepsilon_i)^2-\eta^2/4)((v_k-\varepsilon_i)^2-\eta^2/4)}{(v_k^2-\varepsilon_i^2)(v_k^2-v_i^2)} = 0
\end{aligned}
$$

and cancelling common factors, we obtain

$$
\begin{aligned}
&\frac{\eta}{2v_k}\left(v_k\sqrt{\psi^-\phi^-+1}+\xi^- + \frac{\eta}{2}\sqrt{\psi^-\phi^-+1}\right)\left(v_k\sqrt{\psi^+\phi^++1}+\xi^+ + \frac{\eta}{2}\sqrt{\psi^+\phi^++1}\right)\times \\
&\times \prod_{l=1}^{\mathcal{L}} \frac{(v_k-\varepsilon_l-\eta/2)(v_k+\varepsilon_l-\eta/2)}{(v_k-\varepsilon_l)(v_k+\varepsilon_l)} \prod_{i\neq k}^{\mathcal{L}} \frac{(v_k-v_i+\eta)(v_k+v_i+\eta)}{(v_k-v_i)(v_k+v_i)} - \\
&-\frac{\eta}{2v_k}\left(v_k\sqrt{\psi^-\phi^-+1}-\xi^- - \frac{\eta}{2}\sqrt{\psi^-\phi^-+1}\right)\left(v_k\sqrt{\psi^+\phi^++1}-\xi^+ - \frac{\eta}{2}\sqrt{\psi^+\phi^++1}\right)\times \\
&\times \prod_{l=1}^{\mathcal{L}} \frac{(v_k-\varepsilon_l+\eta/2)(v_k+\varepsilon_l+\eta/2)}{(v_k-\varepsilon_l)(v_k+\varepsilon_l)} \prod_{i\neq k}^{\mathcal{L}} \frac{(v_k-v_i-\eta)(v_k+v_i-\eta)}{(v_k-v_i)(v_k+v_i)} + \\
&+\frac{1}{4}\left((\psi^-\phi^+ + \phi^-\psi^+) + 2 - 2\sqrt{\psi^-\phi^- + 1}\sqrt{\psi^+\phi^+ + 1}\right)\times \\
&\times \prod_{l=1}^{\mathcal{L}} \frac{(v_k+\varepsilon_l-\eta/2)(v_k+\varepsilon_l+\eta/2)(v_k-\varepsilon_l-\eta/2)(v_k-\varepsilon_l+\eta/2)}{(v_k-\varepsilon_l)(v_k+\varepsilon_l)} \prod_{i\neq k}^{\mathcal{L}} \frac{1}{(v_k^2-v_i^2)} = 0.
\end{aligned}
$$

And finally,

$$
\begin{aligned}
&\frac{2\eta}{v_k}\left(v_k\sqrt{\psi^-\phi^-+1}+\xi^- + \frac{\eta}{2}\sqrt{\psi^-\phi^-+1}\right)\left(v_k\sqrt{\psi^+\phi^++1}+\xi^+ + \frac{\eta}{2}\sqrt{\psi^+\phi^++1}\right)\times \\
&\times \prod_{l=1}^{\mathcal{L}} \frac{1}{(v_k-\varepsilon_l+\eta/2)(v_k+\varepsilon_l+\eta/2)} \prod_{i\neq k}^{\mathcal{L}} (v_k-v_i+\eta)(v_k+v_i+\eta) - \\
&-\frac{2\eta}{v_k}\left(v_k\sqrt{\psi^-\phi^-+1}-\xi^- - \frac{\eta}{2}\sqrt{\psi^-\phi^-+1}\right)\left(v_k\sqrt{\psi^+\phi^++1}-\xi^+ - \frac{\eta}{2}\sqrt{\psi^+\phi^++1}\right)\times \\
&\times \prod_{l=1}^{\mathcal{L}} \frac{1}{(v_k-\varepsilon_l-\eta/2)(v_k+\varepsilon_l-\eta/2)} \prod_{i\neq k}^{\mathcal{L}} (v_k-v_i-\eta)(v_k+v_i-\eta) + \\
&+\left((\psi^-\phi^+ + \phi^-\psi^+) + 2 - 2\sqrt{\psi^-\phi^- + 1}\sqrt{\psi^+\phi^+ + 1}\right) = 0. \qquad (4.19)
\end{aligned}
$$

Remark 4.9. *One can also compute the BAE from*

$$
\lim_{u\leftarrow -v_k} (u+v_k)\Lambda(u) = 0.
$$

One can check that this gives the same expression (4.19).

4.3.4 Quasi-classical limit of the Bethe Ansatz Equations

Let us expand the BAE (4.19) in the powers of η. Start with

$$\prod_{l=1}^{\mathcal{L}} \frac{1}{(v_k-\varepsilon_l+\eta/2)(v_k+\varepsilon_l+\eta/2)} = \prod_{l=1}^{\mathcal{L}} \frac{1}{v_k^2-\varepsilon_l^2}\left(1-\frac{\eta}{2}\sum_{l=1}^{\mathcal{L}}\left(\frac{1}{v_k-\varepsilon_l}+\frac{1}{v_k+\varepsilon_l}\right)\right)+\mathcal{O}(\eta^2),$$

$$\prod_{l=1}^{\mathcal{L}} \frac{1}{(v_k-\varepsilon_l-\eta/2)(v_k+\varepsilon_l-\eta/2)} = \prod_{l=1}^{\mathcal{L}} \frac{1}{v_k^2-\varepsilon_l^2}\left(1+\frac{\eta}{2}\sum_{l=1}^{\mathcal{L}}\left(\frac{1}{v_k-\varepsilon_l}+\frac{1}{v_k+\varepsilon_l}\right)\right)+\mathcal{O}(\eta^2),$$

$$\prod_{i\neq k}^{\mathcal{L}}(v_k-v_i+\eta)(v_k+v_i+\eta) = \prod_{i\neq k}^{\mathcal{L}}(v_k^2-v_i^2)\left(1+\eta\sum_{i\neq k}^{\mathcal{L}}\left(\frac{1}{v_k-v_i}+\frac{1}{v_k+v_i}\right)\right)+\mathcal{O}(\eta^2),$$

$$\prod_{i\neq k}^{\mathcal{L}}(v_k-v_i-\eta)(v_k+v_i-\eta) = \prod_{i\neq k}^{\mathcal{L}}(v_k^2-v_i^2)\left(1-\eta\sum_{i\neq k}^{\mathcal{L}}\left(\frac{1}{v_k-v_i}+\frac{1}{v_k+v_i}\right)\right)+\mathcal{O}(\eta^2).$$

Thus,

$$\prod_{l=1}^{\mathcal{L}} \frac{1}{(v_k-\varepsilon_l+\eta/2)(v_k+\varepsilon_l+\eta/2)}\prod_{i\neq k}^{\mathcal{L}}(v_k-v_i+\eta)(v_k+v_i+\eta) =$$
$$= \prod_{l=1}^{\mathcal{L}} \frac{1}{v_k^2-\varepsilon_l^2}\prod_{i\neq k}^{\mathcal{L}}(v_k^2-v_i^2)\left(1+\eta\left(\sum_{i\neq k}^{\mathcal{L}}\frac{2v_k}{v_k^2-v_i^2}-\sum_{l=1}^{\mathcal{L}}\frac{v_k}{v_k^2-\varepsilon_l^2}\right)\right)+\mathcal{O}(\eta^2),$$

$$\prod_{l=1}^{\mathcal{L}} \frac{1}{(v_k-\varepsilon_l-\eta/2)(v_k+\varepsilon_l-\eta/2)}\prod_{i\neq k}^{\mathcal{L}}(v_k-v_i-\eta)(v_k+v_i-\eta) =$$
$$= \prod_{l=1}^{\mathcal{L}} \frac{1}{v_k^2-\varepsilon_l^2}\prod_{i\neq k}^{\mathcal{L}}(v_k^2-v_i^2)\left(1-\eta\left(\sum_{i\neq k}^{\mathcal{L}}\frac{2v_k}{v_k^2-v_i^2}-\sum_{l=1}^{\mathcal{L}}\frac{v_k}{v_k^2-\varepsilon_l^2}\right)\right)+\mathcal{O}(\eta^2).$$

One can check that the first order contribution (i.e. first order in powers of η) from the third term in the sum on the left hand side of (4.19) is zero:

$$\left((\psi^-\phi^+ + \phi^-\psi^+) + 2 - 2\sqrt{\psi^-\phi^- + 1}\sqrt{\psi^+\phi^+ + 1}\right) =$$
$$= \big((\psi+\eta\delta)(\phi+\eta\lambda) + (\phi+\eta\mu)(\psi+\eta\gamma)\big) + 2 -$$
$$- 2(\psi\phi+1)\left(1+\frac{\eta}{2}\frac{\mu\psi+\delta\phi}{\psi\phi+1}\right)\left(1+\frac{\eta}{2}\frac{\lambda\psi+\gamma\phi}{\psi\phi+1}\right)+\mathcal{O}(\eta^2) =$$
$$= \big(2\psi\phi + \eta(\lambda\psi+\delta\phi) + \eta(\mu\psi+\gamma\phi)\big) + 2 - 2(\psi\phi+1) -$$

$$- 2(\psi\phi+1)\frac{\eta}{2}\frac{(\lambda+\mu)\psi+(\delta+\gamma)\phi}{\psi\phi+1}+\mathcal{O}(\eta^2)=\mathcal{O}(\eta^2).$$

The first order contribution from the other two terms in (4.19) also gives zero:

$$\frac{2\eta}{v_k}\left(v_k\sqrt{\psi\phi+1}-\xi\right)\left(v_k\sqrt{\psi\phi+1}+\xi\right)\prod_{l=1}^{\mathcal{L}}\frac{1}{v_k^2-\varepsilon_l^2}\prod_{i\neq k}^{\mathcal{L}}(v_k^2-v_i^2)-$$
$$-\frac{2\eta}{v_k}\left(v_k\sqrt{\psi\phi+1}+\xi\right)\left(v_k\sqrt{\psi\phi+1}-\xi\right)\prod_{l=1}^{\mathcal{L}}\frac{1}{v_k^2-\varepsilon_l^2}\prod_{i\neq k}^{\mathcal{L}}(v_k^2-v_i^2)=0.$$

Thus we have to expand the BAE up to second order. Start with

$$\sqrt{\psi^-\phi^-+1}=\sqrt{\psi\phi+1}\left(1+\frac{1}{2}\frac{\eta(\mu\psi+\delta\phi)+\eta^2\mu\delta}{\psi\phi+1}-\frac{1}{8}\frac{\eta^2(\mu\psi+\delta\phi)^2}{(\psi\phi+1)^2}\right)+\mathcal{O}(\eta^3),$$
$$\sqrt{\psi^+\phi^++1}=\sqrt{\psi\phi+1}\left(1+\frac{1}{2}\frac{\eta(\lambda\psi+\gamma\phi)+\eta^2\lambda\gamma}{\psi\phi+1}-\frac{1}{8}\frac{\eta^2(\lambda\psi+\gamma\phi)^2}{(\psi\phi+1)^2}\right)+\mathcal{O}(\eta^3).$$

Using this, we now calculate the η^2 contribution from the third term of (4.19):

$$\delta\lambda+\mu\gamma-2(\psi\phi+1)\left[\frac{1}{4}\frac{(\mu\psi+\delta\phi)(\lambda\psi+\gamma\phi)}{(\psi\phi+1)^2}+\frac{1}{2}\frac{\mu\delta}{\psi\phi+1}-\frac{1}{8}\frac{(\mu\psi+\delta\phi)^2}{(\psi\phi+1)^2}+\right.$$
$$\left.+\frac{1}{2}\frac{\lambda\gamma}{\psi\phi+1}-\frac{1}{8}\frac{(\lambda\psi+\gamma\phi)^2}{(\psi\phi+1)^2}\right]=$$
$$=\frac{\left((\lambda-\mu)\psi+(\gamma-\delta)\phi\right)^2}{4(\psi\phi+1)}-(\gamma-\delta)(\lambda-\mu).$$

Consider the first and the second terms in (4.19):

$$\frac{2\eta}{v_k}\left(v_k\sqrt{\psi\phi+1}\left(1+\frac{\eta}{2}\frac{\mu\psi+\delta\phi}{\psi\phi+1}\right)-\xi+\eta\beta+\frac{\eta}{2}\sqrt{\psi\phi+1}\right)\times$$
$$\times\left(v_k\sqrt{\psi\phi+1}\left(1+\frac{\eta}{2}\frac{\lambda\psi+\gamma\phi}{\psi\phi+1}\right)+\xi+\eta\alpha+\frac{\eta}{2}\sqrt{\psi\phi+1}\right)\times$$
$$\times\prod_{l=1}^{\mathcal{L}}\frac{1}{v_k^2-\varepsilon_l^2}\prod_{i\neq k}^{\mathcal{L}}(v_k^2-v_i^2)\left(1+\eta\left(\sum_{i\neq k}^{\mathcal{L}}\frac{2v_k}{v_k^2-v_i^2}-\sum_{l=1}^{\mathcal{L}}\frac{v_k}{v_k^2-\varepsilon_l^2}\right)\right)-$$
$$-\frac{2\eta}{v_k}\left(v_k\sqrt{\psi\phi+1}\left(1+\frac{\eta}{2}\frac{\mu\psi+\delta\phi}{\psi\phi+1}\right)+\xi-\eta\beta-\frac{\eta}{2}\sqrt{\psi\phi+1}\right)\times$$
$$\times\left(v_k\sqrt{\psi\phi+1}\left(1+\frac{\eta}{2}\frac{\lambda\psi+\gamma\phi}{\psi\phi+1}\right)-\xi-\eta\alpha-\frac{\eta}{2}\sqrt{\psi\phi+1}\right)\times$$

$$\times \prod_{l=1}^{\mathcal{L}} \frac{1}{v_k^2 - \varepsilon_l^2} \prod_{i\neq k}^{\mathcal{L}} (v_k^2 - v_i^2) \left(1 - \eta \left(\sum_{i\neq k}^{\mathcal{L}} \frac{2v_k}{v_k^2 - v_i^2} - \sum_{l=1}^{\mathcal{L}} \frac{v_k}{v_k^2 - \varepsilon_l^2}\right)\right).$$

The first term gives the η^2 contribution

$$\frac{2}{v_k} \prod_{l=1}^{\mathcal{L}} \frac{1}{v_k^2 - \varepsilon_l^2} \prod_{i\neq k}^{\mathcal{L}} (v_k^2 - v_i^2) \Bigg[\left(v_k\sqrt{\psi\phi+1} - \xi\right) \left(\frac{v_k}{2}\frac{\lambda\psi + \gamma\phi}{\sqrt{\psi\phi+1}} + \alpha + \frac{\sqrt{\psi\phi+1}}{2}\right) + \\ + \left(v_k\sqrt{\psi\phi+1} + \xi\right) \left(\frac{v_k}{2}\frac{\mu\psi + \delta\phi}{\sqrt{\psi\phi+1}} + \beta + \frac{\sqrt{\psi\phi+1}}{2}\right) + \\ + \left(v_k^2(\psi\phi+1) - \xi^2\right) \left(\sum_{i\neq k}^{\mathcal{L}} \frac{2v_k}{v_k^2 - v_i^2} - \sum_{l=1}^{\mathcal{L}} \frac{v_k}{v_k^2 - \varepsilon_l^2}\right) \Bigg],$$

and the second term gives the η^2 contribution

$$\frac{2}{v_k} \prod_{l=1}^{\mathcal{L}} \frac{1}{v_k^2 - \varepsilon_l^2} \prod_{i\neq k}^{\mathcal{L}} (v_k^2 - v_i^2) \Bigg[\left(v_k\sqrt{\psi\phi+1} + \xi\right) \left(\frac{v_k}{2}\frac{\lambda\psi + \gamma\phi}{\sqrt{\psi\phi+1}} - \alpha - \frac{\sqrt{\psi\phi+1}}{2}\right) + \\ + \left(v_k\sqrt{\psi\phi+1} - \xi\right) \left(\frac{v_k}{2}\frac{\mu\psi + \delta\phi}{\sqrt{\psi\phi+1}} - \beta - \frac{\sqrt{\psi\phi+1}}{2}\right) - \\ - \left(v_k^2(\psi\phi+1) - \xi^2\right) \left(\sum_{i\neq k}^{\mathcal{L}} \frac{2v_k}{v_k^2 - v_i^2} - \sum_{l=1}^{\mathcal{L}} \frac{v_k}{v_k^2 - \varepsilon_l^2}\right) \Bigg].$$

Summing up all terms we obtain

$$2 \prod_{l=1}^{\mathcal{L}} \frac{1}{v_k^2 - \varepsilon_l^2} \prod_{i\neq k}^{\mathcal{L}} (v_k^2 - v_i^2) \Bigg[2(\alpha+\beta)\sqrt{\psi\phi+1} + 2(\psi\phi+1) - \frac{\xi\big((\lambda-\mu)\psi + (\gamma-\delta)\phi\big)}{\sqrt{\psi\phi+1}} + \\ + 2\left(v_k^2(\psi\phi+1) - \xi^2\right) \left(\sum_{i\neq k}^{\mathcal{L}} \frac{2}{v_k^2 - v_i^2} - \sum_{l=1}^{\mathcal{L}} \frac{1}{v_k^2 - \varepsilon_l^2}\right) \Bigg] + \\ + \frac{\big((\lambda-\mu)\psi + (\gamma-\delta)\phi\big)^2}{4(\psi\phi+1)} - (\gamma-\delta)(\lambda-\mu) = 0.$$

Thus, we obtain the following BAE in the quasi-classical limit:

$$\begin{aligned}(\alpha+\beta)\sqrt{\psi\phi+1}+(\psi\phi+1)-\frac{\xi\left((\lambda-\mu)\psi+(\gamma-\delta)\phi\right)}{2\sqrt{\psi\phi+1}}+\\ +\left(v_k^2(\psi\phi+1)-\xi^2\right)\left(\sum_{i\neq k}^{\mathcal{L}}\frac{2}{v_k^2-v_i^2}-\sum_{l=1}^{\mathcal{L}}\frac{1}{v_k^2-\varepsilon_l^2}\right)=\\ =\frac{1}{4}\left((\gamma-\delta)(\lambda-\mu)-\frac{((\lambda-\mu)\psi+(\gamma-\delta)\phi)^2}{4(\psi\phi+1)}\right)\frac{\prod_{l=1}^{\mathcal{L}}(v_k^2-\varepsilon_l^2)}{\prod_{i\neq k}^{\mathcal{L}}(v_k^2-v_i^2)}.\end{aligned} \tag{4.20}$$

Remark 4.10. *By setting $\beta=\psi=\phi=\delta=\mu=\xi=0$ in (4.20) we deduce the Bethe roots $\{v_k : k=1,\ldots,\mathcal{L}\}$ appearing in (4.17) satisfy the BAE*

$$\frac{\alpha+1}{v_k^2}+\sum_{i\neq k}^{\mathcal{L}}\frac{2}{v_k^2-v_i^2}-\sum_{l=1}^{\mathcal{L}}\frac{1}{v_k^2-\varepsilon_l^2}=\frac{\gamma\lambda}{4v_k^2}\frac{\prod_{l=1}^{\mathcal{L}}(v_k^2-\varepsilon_l^2)}{\prod_{i\neq k}^{\mathcal{L}}(v_k^2-v_i^2)}. \tag{4.21}$$

4.3.5 Eigenvalues of the Hamiltonian

Recall λ_j^*, given in (4.17), is the eigenvalue of the conserved operator τ_j^* given in (4.10). To compute the eigenvalue of the Hamiltonian (4.11) consider

$$\begin{aligned}\sum_{j=1}^{\mathcal{L}}\varepsilon_j^{-2}\lambda_j^*&=\sum_{j,k:k\neq j}^{\mathcal{L}}\frac{1}{\varepsilon_j^2-\varepsilon_k^2}-\sum_{i=1}^{\mathcal{L}}\sum_{j=1}^{\mathcal{L}}\frac{2}{\varepsilon_j^2-v_i^2}-\alpha\sum_{j=1}^{\mathcal{L}}\varepsilon_j^{-2}=\\ &=\sum_{i,j=1}^{\mathcal{L}}\frac{2}{v_i^2-\varepsilon_j^2}-\alpha\sum_{j=1}^{\mathcal{L}}\varepsilon_j^{-2}.\end{aligned}$$

From the BAE (4.21) we find

$$\sum_{j=1}^{\mathcal{L}}\frac{1}{v_i^2-\varepsilon_j^2}=\frac{\alpha+1}{v_i^2}+\sum_{k\neq i}^{\mathcal{L}}\frac{2}{v_i^2-v_k^2}-\frac{\gamma\lambda}{4v_i^2}\frac{\prod_{j=1}^{\mathcal{L}}(v_i^2-\varepsilon_j^2)}{\prod_{k\neq i}^{\mathcal{L}}(v_i^2-v_k^2)}.$$

Thus,

$$\begin{aligned}\sum_{i=1}^{\mathcal{L}}\sum_{j=1}^{\mathcal{L}}\frac{2}{v_i^2-\varepsilon_j^2}&=2(\alpha+1)\sum_{i=1}^{\mathcal{L}}v_i^{-2}+\sum_{i=1}^{\mathcal{L}}\sum_{k\neq i}^{\mathcal{L}}\frac{4}{v_i^2-v_k^2}-\frac{\gamma\lambda}{2}\sum_{i=1}^{\mathcal{L}}\frac{1}{v_i^2}\frac{\prod_{j=1}^{\mathcal{L}}(v_i^2-\varepsilon_j^2)}{\prod_{k\neq i}^{\mathcal{L}}(v_i^2-v_k^2)}=\\ &=2(\alpha+1)\sum_{i=1}^{\mathcal{L}}v_i^{-2}-\frac{\gamma\lambda}{2}\sum_{i=1}^{\mathcal{L}}\frac{1}{v_i^2}\frac{\prod_{j=1}^{\mathcal{L}}(v_i^2-\varepsilon_j^2)}{\prod_{k\neq i}^{\mathcal{L}}(v_i^2-v_k^2)}.\end{aligned}$$

This leads to

$$\sum_{j=1}^{\mathcal{L}} \varepsilon_j^{-2}\lambda_j^* = 2(\alpha+1)\sum_{i=1}^{\mathcal{L}} v_i^{-2} - \frac{\gamma\lambda}{2}\sum_{i=1}^{\mathcal{L}} \frac{1}{v_i^2}\frac{\prod_{j=1}^{\mathcal{L}}(v_i^2-\varepsilon_j^2)}{\prod_{k\neq i}^{\mathcal{L}}(v_i^2-v_k^2)} - \alpha\sum_{j=1}^{\mathcal{L}}\varepsilon_j^{-2}.$$

Implementing the change of variables $z_j = \varepsilon_j^{-1},\ \ y_i = v_i^{-2}$ and setting $\lambda = -\gamma$ we obtain the expression

$$E = (1+G)\sum_{i=1}^{\mathcal{L}} y_i - \frac{1}{2}\sum_{k=1}^{\mathcal{L}} z_k^2 + \frac{\Gamma^2}{G}\sum_{i=1}^{\mathcal{L}} \frac{\prod_{j=1}^{\mathcal{L}}(1-y_i z_j^{-2})}{\prod_{k\neq i}^{\mathcal{L}}(1-y_i y_k^{-1})} \tag{4.22}$$

for the eigenvalues of the Hamiltonian (4.14) subject to the BAE obtained from (4.21):

$$1 + G^{-1} + \sum_{i\neq k}^{\mathcal{L}} \frac{2y_i}{y_i - y_k} + \sum_{l=1}^{\mathcal{L}} \frac{z_l^2}{y_k - z_l^2} = -\frac{\Gamma^2}{G^2 y_k}\frac{\prod_{l=1}^{\mathcal{L}}(1-y_k z_l^{-2})}{\prod_{i\neq k}^{\mathcal{L}}(1-y_k y_i^{-1})}. \tag{4.23}$$

4.4 Summary

In this chapter we have studied the open, rational Richardson–Gaudin model with off-diagonal boundary, i.e., obtained in the quasi-classical limit from the BQISM with rational off-diagonal reflection matrices. Assuming one of the K-matrices being diagonal (which can almost always be achieved by a basis transformation), we have arrived at the expression (4.10) for the conserved operators. Next, we have constructed the Hamiltonian (4.14) (which is equivalent to (4.12) in the spin operator formalism) as a linear combination of these operators. Thus, we have shown that the Hamiltonian (4.12), describing a $p+ip$ pairing model interacting with its environment, is an integrable model. Finally, by applying the off-diagonal Bethe Ansatz, we found that the energies of (4.14) are given by (4.22) subject to the BAE (4.23), and the eigenvalues of the conserved operators (4.10) are given by (4.17).

The new integrable Hamiltonian (4.14) was recently applied by Claeys et al [CBN16] to model a two-dimensional $p_x + ip_y$ superfluid interacting with an environment. The exact Richardson–Gaudin wave function was presented and the BAE (4.23) were re-derived using an alternative algebraic Bethe Ansatz [TF14]. Derivation of the exact eigenvalues of the conserved operators (4.10), exact solution of the BAE and calculation of the exact correlation functions were also discussed. These exact results were compared with the mean field theory approach.

Chapter 5

Trigonometric Richardson–Gaudin models with off-diagonal reflection matrices

In this chapter we consider the most challenging case of Richardson–Gaudin models from the BQISM so far, based on the trigonometric off-diagonal reflection matrices. The following K-matrices [CYSW13c] satisfy the reflection equations (2.32) together with the trigonometric R-matrix (2.2):

$$\check{K}^-(u) = \begin{pmatrix} \check{K}^-_{11}(u) & \check{K}^-_{12}(u) \\ \check{K}^-_{21}(u) & \check{K}^-_{22}(u) \end{pmatrix},$$

where

$$\begin{aligned} \check{K}^-_{11}(u) &= 2\left(\sinh\alpha_- \cosh\beta_- \cosh u + \cosh\alpha_- \sinh\beta_- \sinh u\right), \\ \check{K}^-_{22}(u) &= 2\left(\sinh\alpha_- \cosh\beta_- \cosh u - \cosh\alpha_- \sinh\beta_- \sinh u\right), \\ \check{K}^-_{12}(u) &= e^{\theta_-}\sinh(2u),\ \check{K}^-_{21}(u) = e^{-\theta_-}\sinh(2u), \end{aligned}$$

and[1]

$$\check{K}^+(u) = -\,\check{K}^-(-u-\eta)\big|_{(\alpha_-,\beta_-,\theta_-)\mapsto(-\alpha_+,-\beta_+,\theta_+)}\,.$$

We make the usual change of variable $u \mapsto u - \eta/2$, so that the K-matrices are now given by

$$K^-(u) = \check{K}^-(u-\eta/2) = \begin{pmatrix} K^-_{11}(u) & K^-_{12}(u) \\ K^-_{21}(u) & K^-_{22}(u) \end{pmatrix}, \tag{5.1a}$$

$$K^+(u) = \check{K}^+(u-\eta/2) = \begin{pmatrix} K^+_{11}(u) & K^+_{12}(u) \\ K^+_{21}(u) & K^+_{22}(u) \end{pmatrix}, \tag{5.1b}$$

[1] Note that our $\check{K}^+(u)$ differs by the minus sign from the one in [CYSW13c]. We choose this convention in order to make the K-matrices consistent with those from previous chapters.

where

$$
\begin{aligned}
K_{11}^{-}(u) &= 2\big(\sinh\alpha_{-}\cosh\beta_{-}\cosh(u-\eta/2)+\cosh\alpha_{-}\sinh\beta_{-}\sinh(u-\eta/2)\big),\\
K_{22}^{-}(u) &= 2\big(\sinh\alpha_{-}\cosh\beta_{-}\cosh(u-\eta/2)-\cosh\alpha_{-}\sinh\beta_{-}\sinh(u-\eta/2)\big),\\
K_{12}^{-}(u) &= e^{\theta_{-}}\sinh(2u-\eta),\ K_{21}^{-}(u)=\ e^{-\theta_{-}}\sinh(2u-\eta),
\end{aligned}
$$

and

$$
\begin{aligned}
K_{11}^{+}(u) &= 2\big(\sinh\alpha_{+}\cosh\beta_{+}\cosh(u+\eta/2)-\cosh\alpha_{+}\sinh\beta_{+}\sinh(u+\eta/2)\big),\\
K_{22}^{+}(u) &= 2\big(\sinh\alpha_{+}\cosh\beta_{+}\cosh(u+\eta/2)+\cosh\alpha_{+}\sinh\beta_{+}\sinh(u+\eta/2)\big),\\
K_{12}^{+}(u) &= e^{\theta_{+}}\sinh(2u+\eta),\ K_{21}^{+}(u)=e^{-\theta_{+}}\sinh(2u+\eta).
\end{aligned}
$$

Consider the transfer matrix of the form (2.44)

$$
t(u)=\operatorname{tr}_a\left(K_a^{+}(u)L_{a\mathcal{L}}(u-\varepsilon_{\mathcal{L}})\cdots L_{a1}(u-\varepsilon_1)K_a^{-}(u)L_{a1}(u+\varepsilon_1)\cdots L_{a\mathcal{L}}(u+\varepsilon_{\mathcal{L}})\right) \quad (5.2)
$$

with the K-matrices given by (5.1) and the trigonometric Lax operator (2.17). The quasi-classical limit of this transfer matrix leads to the open, trigonometric Richardson–Gaudin model with off-diagonal boundary.

The plan for this chapter is the following. First of all, in Sections 5.1 and 5.2 we describe the diagonal and rational limits of the K-matrices (5.1), leading to the constructions from Chapters 3 and 4 respectively. In Section 5.3 we discuss how to take the quasi-classical limit in this case and in Section 5.4 we derive the conserved operators as the quasi-classical limit of the transfer matrix (5.2). Next, we consider the second family of the conserved operators and prove that it is equivalent to the first one (see Proposition 5.4 below). Our construction and, in particular, the expression for the conserved operators depends on several free parameters. In Section 5.6 we consider a special case obtained by setting some of the parameters to zero. This simplifies the conserved operators to the expression (5.8) which is easier to analyse. We show that this expression is different from all conserved operators we have seen in previous chapters. In fact, we notice a certain similarity with the elliptic Gaudin model [ED15]. We close the chapter with a discussion of future research possibilities.

5.1 Diagonal limit

The diagonal K-matrices (2.42) can be obtained as follows from (5.1). Rescale (5.1a) by $e^{-\beta_-}$ and send $\beta_- \to \infty$. Then, using

$$e^{-\beta_-}\sinh\beta_- \xrightarrow{\beta_-\to\infty} 1/2, \quad e^{-\beta_-}\cosh\beta_- \xrightarrow{\beta_-\to\infty} 1/2,$$

we obtain

$$\begin{aligned} e^{-\beta_-}K^-(u) \to \operatorname{diag}\big(&\sinh\alpha_-\cosh(u-\eta/2)+\cosh\alpha_-\sinh(u-\eta/2),\\ &\sinh\alpha_-\cosh(u-\eta/2)-\cosh\alpha_-\sinh(u-\eta/2)\big) = \\ =\operatorname{diag}\big(&\sinh(\alpha_-+u-\eta/2),\sinh(\alpha_--u+\eta/2)\big), \end{aligned}$$

which is equal to the diagonal K^--matrix (2.42a) with $\xi^- = \alpha_-$.

Analogously, rescale (5.1b) by e^{β_+} and send $\beta_+ \to -\infty$. Using

$$e^{\beta_+}\sinh\beta_+ \xrightarrow{\beta_+\to-\infty} -1/2, \quad e^{\beta_+}\cosh\beta_+ \xrightarrow{\beta_+\to-\infty} 1/2,$$

we obtain

$$\begin{aligned} e^{\beta_+}K^+(u) \to \operatorname{diag}\big(&\sinh\alpha_+\cosh(u+\eta/2)+\cosh\alpha_+\sinh(u+\eta/2),\\ &\sinh\alpha_+\cosh(u+\eta/2)-\cosh\alpha_+\sinh(u+\eta/2)\big) = \\ =\operatorname{diag}\big(&\sinh(\alpha_++u+\eta/2),\sinh(\alpha_+-u-\eta/2)\big), \end{aligned}$$

which is equal to the diagonal K^+-matrix (2.42b) with $\xi^+ = \alpha_+$.

5.2 Rational limit

Now let us explain how to obtain the rational off-diagonal K-matrices (4.1) from (5.1). First of all, let us introduce an additional parameter ν as follows:

$$\alpha_- \mapsto \nu\alpha_-,\ \alpha_+ \mapsto \nu\alpha_+,\ u+\eta/2 \mapsto \nu(u+\eta/2).$$

Divide the K-matrices (5.1) by ν and use

$$\lim_{\nu\to 0}\frac{\sinh(\nu x)}{\nu} = x$$

to obtain

$$K^-_{11}(u) = 2\big(\alpha_-\cosh\beta_- + \sinh\beta_-(u-\eta/2)\big) = 2\sinh\beta_-(\alpha_-\coth\beta_- + u - \eta/2),$$
$$K^-_{22}(u) = 2\big(\alpha_-\cosh\beta_- - \sinh\beta_-(u-\eta/2)\big) = 2\sinh\beta_-(\alpha_-\coth\beta_- - u + \eta/2),$$
$$K^-_{12}(u) = 2e^{\theta_-}(u-\eta/2),\ K^-_{21}(u) = 2e^{-\theta_-}(u-\eta/2).$$

Thus, in this limit (5.1a) will reduce to (4.1a) up to a scalar multiple:

$$K^-(u) = 2\sinh\beta_-\begin{pmatrix}\alpha_-\coth\beta_- + u - \eta/2 & \frac{e^{\theta_-}}{\sinh\beta_-}(u-\eta/2)\\ \frac{e^{-\theta_-}}{\sinh\beta_-}(u-\eta/2) & \alpha_-\coth\beta_- - u + \eta/2\end{pmatrix} =$$
$$= 2\sinh\beta_-\begin{pmatrix}\xi^- + u - \eta/2 & \psi^-(u-\eta/2)\\ \phi^-(u-\eta/2) & \xi^- - u + \eta/2\end{pmatrix} \tag{5.3}$$

with

$$\xi^- = \alpha_-\coth\beta_-,\ \psi^- = \frac{e^{\theta_-}}{\sinh\beta_-},\ \phi^- = \frac{e^{-\theta_-}}{\sinh\beta_-}.$$

And, analogously,

$$K^+_{11}(u) = 2\big(\alpha_+\cosh\beta_+ - \sinh\beta_+(u+\eta/2)\big) = -2\sinh\beta_+(-\alpha_+\coth\beta_+ + u + \eta/2),$$
$$K^+_{22}(u) = 2\big(\alpha_+\cosh\beta_+ + \sinh\beta_+(u+\eta/2)\big) = -2\sinh\beta_+(-\alpha_+\coth\beta_+ - u - \eta/2),$$
$$K^+_{12}(u) = 2e^{\theta_+}(u+\eta/2),\ K^+_{21}(u) = 2e^{-\theta^+}(u+\eta/2).$$

Thus, (5.1b) will reduce to (4.1b) up to a scalar multiple:

$$K^+(u) = -2\sinh\beta_+\begin{pmatrix}-\alpha_+\coth\beta_+ + u + \eta/2 & -\frac{e^{\theta_+}}{\sinh\beta_+}(u+\eta/2)\\ -\frac{e^{-\theta_+}}{\sinh\beta_+}(u+\eta/2) & -\alpha_+\coth\beta_+ - u - \eta/2\end{pmatrix} =$$
$$= -2\sinh\beta_+\begin{pmatrix}\xi^+ + u + \eta/2 & \psi^+(u+\eta/2)\\ \phi^+(u+\eta/2) & \xi^+ - u - \eta/2\end{pmatrix} \tag{5.4}$$

with

$$\xi^+ = -\alpha_+\coth\beta_+,\ \psi^+ = -\frac{e^{\theta_+}}{\sinh\beta_+},\ \phi^+ = -\frac{e^{-\theta_+}}{\sinh\beta_+}.$$

Remark 5.1. *For completeness of the picture let us also check the diagonal limit.*

- *Multiply (5.3) by $e^{-\beta_-}$ and consider $\beta_-\to\infty$. Then*

$$K^-(u) \to 2\begin{pmatrix}\frac{1}{2}\alpha_- + \frac{1}{2}(u-\eta/2) & 0\\ 0 & \frac{1}{2}\alpha_- - \frac{1}{2}(u-\eta/2)\end{pmatrix} =$$

$$= \begin{pmatrix} \alpha_- + u - \eta/2 & 0 \\ 0 & \alpha_- - u + \eta/2 \end{pmatrix}.$$

- *Multiply (5.4) by e^{β_+} and consider $\beta_+ \to -\infty$. Then*

$$K^+(u) \to -2 \begin{pmatrix} -\frac{1}{2}\alpha_+ - \frac{1}{2}(u+\eta/2) & 0 \\ 0 & -\frac{1}{2}\alpha_+ + \frac{1}{2}(u+\eta/2) \end{pmatrix} =$$
$$= \begin{pmatrix} \alpha_+ + u + \eta/2 & 0 \\ 0 & \alpha_+ - u - \eta/2 \end{pmatrix}.$$

These agree (up to a rescaling) with the rational limit (2.57) of (2.42), where we identify $\xi^- = \alpha_-$, $\xi^+ = \alpha_+$.

5.3 Quasi-classical limit

Like in the rational case (Section 4.2) we require that the K-matrices (5.1) satisfy the condition (4.3):

$$\lim_{\eta\to 0}\{K^+(u)K^-(u)\} \propto I.$$

Let us assume the following dependence of parameters on η:

$$\begin{aligned} \alpha_+ = \xi + \eta\alpha, \quad \beta_+ = \zeta + \eta\gamma, \quad \theta_+ = \theta + \eta t, \\ \alpha_- = -\xi + \eta\beta, \quad \beta_- = -\zeta + \eta\delta, \quad \theta_- = \theta + \eta s, \end{aligned} \tag{5.5}$$

where $\xi,\ \zeta,\ \alpha,\ \beta,\ \gamma,\ \delta,\ \theta,\ t,\ s$ are free complex parameters (there are no constraints between them).

Now consider

$$K^+(u)K^-(u) = \begin{pmatrix} K^+_{11}(u) & K^+_{12}(u) \\ K^+_{21}(u) & K^+_{22}(u) \end{pmatrix} \begin{pmatrix} K^-_{11}(u) & K^-_{12}(u) \\ K^-_{21}(u) & K^-_{22}(u) \end{pmatrix} =$$
$$= \begin{pmatrix} K^+_{11}(u)K^-_{11}(u) + K^+_{12}(u)K^-_{21}(u) & K^+_{11}(u)K^-_{12}(u) + K^+_{12}(u)K^-_{22}(u) \\ K^+_{21}(u)K^-_{11}(u) + K^+_{22}(u)K^-_{21}(u) & K^+_{21}(u)K^-_{12}(u) + K^+_{22}(u)K^-_{22}(u) \end{pmatrix}.$$

Setting $\eta = 0$ we obtain

$$K^+_{21}(u)K^-_{11}(u) + K^+_{22}(u)K^-_{21}(u)\big|_{\eta=0} =$$

$$= 2e^{-\theta}\sinh(2u)\left(-\sinh\xi\cosh\zeta\cosh u - \cosh\xi\sinh\zeta\sinh u\right) +$$
$$+ 2e^{-\theta}\sinh(2u)\left(\sinh\xi\cosh\zeta\cosh u + \cosh\xi\sinh\zeta\sinh u\right) = 0,$$
$$K^+_{11}(u)K^-_{12}(u) + K^+_{12}(u)K^-_{22}(u)\big|_{\eta=0} =$$
$$= 2e^{\theta}\sinh(2u)\left(\sinh\xi\cosh\zeta\cosh u - \cosh\xi\sinh\zeta\sinh u\right) +$$
$$+ 2e^{\theta}\sinh(2u)\left(-\sinh\xi\cosh\zeta\cosh u + \cosh\xi\sinh\zeta\sinh u\right) = 0,$$
$$K^+_{11}(u)K^-_{11}(u) + K^+_{12}(u)K^-_{21}(u)\big|_{\eta=0} =$$
$$= \sinh^2(2u) - 4\big((\sinh\xi\cosh\zeta\cosh u)^2 - (\cosh\xi\sinh\zeta\sinh u)^2\big),$$
$$K^+_{21}(u)K^-_{12}(u) + K^+_{22}(u)K^-_{22}(u)\big|_{\eta=0} =$$
$$= \sinh^2(2u) - 4\big((\sinh\xi\cosh\zeta\cosh u)^2 - (\cosh\xi\sinh\zeta\sinh u)^2\big).$$

Thus, the condition (4.3) holds under assumption (5.5).

Remark 5.2. *To retrieve the diagonal construction from here we send $\zeta \to -\infty$ and set $\gamma = \delta = 0$. This leads to $\beta_- \to \infty$ and $\beta_+ \to -\infty$. Thus, by Section 5.1, we have*

$$e^{\zeta}K^-(u) \xrightarrow{\zeta\to-\infty} (2.42a) \quad \textit{with} \quad \xi^- = -\xi + \eta\beta,$$
$$e^{\zeta}K^+(u) \xrightarrow{\zeta\to-\infty} (2.42b) \quad \textit{with} \quad \xi^+ = \xi + \eta\alpha.$$

This agrees with the quasi-classical expansion (3.3) in the diagonal case.

5.4 The first family of conserved operators

The conserved operators τ_j are constructed as before (3.14) from the transfer matrix (5.2) above:

$$\lim_{u\to\varepsilon_j}(u-\varepsilon_j)t(u) = \eta^2\tau_j + \mathcal{O}(\eta^3).$$

Expanding the K-matrices in η as $\eta \to 0$ taking into account (5.5) we obtain

$$K^+(u) = K^+_1(u) + \eta K^+_2(u) + \mathcal{O}(\eta^2), \quad K^-(u) = K^-_1(u) + \eta K^-_2(u) + \mathcal{O}(\eta^2), \tag{5.6}$$

where

$$K^+_1(u) =$$
$$= \begin{pmatrix} 2\left(\sinh\xi\cosh\zeta\cosh u - \cosh\xi\sinh\zeta\sinh u\right) & e^{\theta}\sinh(2u) \\ e^{-\theta}\sinh(2u) & 2\left(\sinh\xi\cosh\zeta\cosh u + \cosh\xi\sinh\zeta\sinh u\right) \end{pmatrix},$$

$$K_2^+(u) =$$

$$= \begin{pmatrix} \begin{array}{c} 2\sinh\xi\sinh\zeta(\gamma\cosh u - \alpha\sinh u) + \\ +2\cosh\xi\cosh\zeta(\alpha\cosh u - \gamma\sinh u) + \\ +(\sinh\xi\cosh\zeta\sinh u - \cosh\xi\sinh\zeta\cosh u) \end{array} & e^{\theta}(t\sinh(2u) + \cosh(2u)) \\ e^{-\theta}(-t\sinh(2u) + \cosh(2u)) & \begin{array}{c} 2\sinh\xi\sinh\zeta(\gamma\cosh u + \alpha\sinh u) + \\ +2\cosh\xi\cosh\zeta(\alpha\cosh u + \gamma\sinh u) + \\ +(\sinh\xi\cosh\zeta\sinh u + \cosh\xi\sinh\zeta\cosh u) \end{array} \end{pmatrix},$$

and

$$K_1^-(u) =$$

$$= \begin{pmatrix} -2\left(\sinh\xi\cosh\zeta\cosh u + \cosh\xi\sinh\zeta\sinh u\right) & e^{\theta}\sinh(2u) \\ e^{-\theta}\sinh(2u) & -2\left(\sinh\xi\cosh\zeta\cosh u - \cosh\xi\sinh\zeta\sinh u\right) \end{pmatrix},$$

$$K_2^-(u) =$$

$$= \begin{pmatrix} \begin{array}{c} 2\sinh\xi\sinh\zeta(\delta\cosh u + \beta\sinh u) + \\ +2\cosh\xi\cosh\zeta(\beta\cosh u + \delta\sinh u) + \\ +(\sinh\xi\cosh\zeta\sinh u + \cosh\xi\sinh\zeta\cosh u) \end{array} & e^{\theta}(s\sinh(2u) - \cosh(2u)) \\ -e^{-\theta}(s\sinh(2u) + \cosh(2u)) & \begin{array}{c} 2\sinh\xi\sinh\zeta(\delta\cosh u - \beta\sinh u) + \\ +2\cosh\xi\cosh\zeta(\beta\cosh u - \delta\sinh u) + \\ +(\sinh\xi\cosh\zeta\sinh u - \cosh\xi\sinh\zeta\cosh u) \end{array} \end{pmatrix}.$$

Substituting (5.6) and (2.24) into (5.2) we obtain the following (the same as in Section 3.2, but with different K-matrices):

$$\begin{aligned} &\lim_{u\to\varepsilon_j}(u-\varepsilon_j)t(u) = \\ &= \eta\,\mathrm{tr}_a\Bigg[K_{1a}^+(\varepsilon_j)\ell_{aj}(0)K_{1a}^-(\varepsilon_j) + \eta\sum_{k=j+1}^{\mathcal{L}}\frac{K_{1a}^+(\varepsilon_j)\ell_{ak}(\varepsilon_j-\varepsilon_k)\ell_{aj}(0)K_{1a}^-(\varepsilon_j)}{\sinh(\varepsilon_j-\varepsilon_k)} + \\ &\qquad + \eta\sum_{k=1}^{j-1}\frac{K_{1a}^+(\varepsilon_j)\ell_{aj}(0)\ell_{ak}(\varepsilon_j-\varepsilon_k)K_{1a}^-(\varepsilon_j)}{\sinh(\varepsilon_j-\varepsilon_k)} + \eta K_{2a}^+(\varepsilon_j)\ell_{aj}(0)K_{1a}^-(\varepsilon_j) + \\ &\qquad + \eta\sum_{k=1}^{\mathcal{L}}\frac{K_{1a}^+(\varepsilon_j)\ell_{aj}(0)K_{1a}^-(\varepsilon_j)\ell_{ak}(\varepsilon_j+\varepsilon_k)}{\sinh(\varepsilon_j+\varepsilon_k)} + \eta K_{1a}^+(\varepsilon_j)\ell_{aj}(0)K_{2a}^-(\varepsilon_j)\Bigg] + \mathcal{O}(\eta^3). \end{aligned}$$

Again, one can check that

$$\mathrm{tr}_a\left(K_{1a}^+(\varepsilon_j)\ell_{aj}(0)K_{1a}^-(\varepsilon_j)\right) = 0,$$
$$\mathrm{tr}_a\left(K_{1a}^+(\varepsilon_j)\ell_{ak}(\varepsilon_j-\varepsilon_k)\ell_{aj}(0)K_{1a}^-(\varepsilon_j)\right) = \mathrm{tr}_a\left(K_{1a}^+(\varepsilon_j)\ell_{aj}(0)\ell_{ak}(\varepsilon_j-\varepsilon_k)K_{1a}^-(\varepsilon_j)\right),$$

which gives

$$\tau_j = \sum_{k\neq j}^{\mathcal{L}} \frac{\mathrm{tr}_a\left(K^+_{1a}(\varepsilon_j)\ell_{ak}(\varepsilon_j-\varepsilon_k)\ell_{aj}(0)K^-_{1a}(\varepsilon_j)\right)}{\sinh(\varepsilon_j-\varepsilon_k)} +$$
$$+\sum_{k=1}^{\mathcal{L}} \frac{\mathrm{tr}_a\left(K^+_{1a}(\varepsilon_j)\ell_{aj}(0)K^-_{1a}(\varepsilon_j)\ell_{ak}(\varepsilon_j+\varepsilon_k)\right)}{\sinh(\varepsilon_j+\varepsilon_k)} +$$
$$+\,\mathrm{tr}_a\left(K^+_{2a}(\varepsilon_j)\ell_{aj}(0)K^-_{1a}(\varepsilon_j)\right) + \mathrm{tr}_a\left(K^+_{1a}(\varepsilon_j)\ell_{aj}(0)K^-_{2a}(\varepsilon_j)\right).$$

In the following we will work with the three parts of the above expression separately and it is convenient to introduce the shorthand notation

$$T_1 \equiv \mathrm{tr}_a\left(K^+_{1a}(\varepsilon_j)\ell_{ak}(\varepsilon_j-\varepsilon_k)\ell_{aj}(0)K^-_{1a}(\varepsilon_j)\right),$$
$$T_2 \equiv \mathrm{tr}_a\left(K^+_{1a}(\varepsilon_j)\ell_{aj}(0)K^-_{1a}(\varepsilon_j)\ell_{ak}(\varepsilon_j+\varepsilon_k)\right),$$
$$T_3 \equiv \mathrm{tr}_a\left(K^+_{2a}(\varepsilon_j)\ell_{aj}(0)K^-_{1a}(\varepsilon_j)\right) + \mathrm{tr}_a\left(K^+_{1a}(\varepsilon_j)\ell_{aj}(0)K^-_{2a}(\varepsilon_j)\right),$$

and, hence,

$$\tau_j = \sum_{k\neq j}^{\mathcal{L}} \frac{T_1}{\sinh(\varepsilon_j-\varepsilon_k)} + \sum_{k=1}^{\mathcal{L}} \frac{T_2}{\sinh(\varepsilon_j+\varepsilon_k)} + T_3. \tag{5.7}$$

Using Maple and simplifying we can calculate the traces T_1, T_2 and T_3 explicitly:

$$T_1 = 4\left(\sinh^2\varepsilon_j + \cosh^2\zeta\right)\left(\cosh^2\varepsilon_j - \cosh^2\xi\right)\left(S^+_jS^-_k + S^-_jS^+_k + 2\cosh(\varepsilon_j-\varepsilon_k)S^z_jS^z_k\right),$$
$$T_2 = -4\left(\sinh\xi\cosh\zeta\cosh\varepsilon_j + \cosh\xi\sinh\zeta\sinh\varepsilon_j\right)^2 S^+_jS^-_k -$$
$$-\,4\left(\sinh\xi\cosh\zeta\cosh\varepsilon_j - \cosh\xi\sinh\zeta\sinh\varepsilon_j\right)^2 S^-_jS^+_k -$$
$$-\,8\cosh(\varepsilon_j+\varepsilon_k)\left(\sinh^2\xi\cosh^2\zeta\cosh^2\varepsilon_j - \cosh^2\xi\sinh^2\zeta\sinh^2\varepsilon_j\right)S^z_jS^z_k -$$
$$-\,2\cosh(\varepsilon_j+\varepsilon_k)\sinh^2(2\varepsilon_j)S^z_jS^z_k -$$
$$-\,4\sinh(2\varepsilon_j)\left(\sinh\xi\cosh\zeta\cosh\varepsilon_j + \cosh\xi\sinh\zeta\sinh\varepsilon_j\right)\times$$
$$\times\left(e^{-\theta}S^z_jS^-_k + \cosh(\varepsilon_j+\varepsilon_k)e^{\theta}S^+_jS^z_k\right) +$$
$$+\,4\sinh(2\varepsilon_j)\left(\sinh\xi\cosh\zeta\cosh\varepsilon_j - \cosh\xi\sinh\zeta\sinh\varepsilon_j\right)\times$$
$$\times\left(e^{\theta}S^z_jS^+_k + \cosh(\varepsilon_j+\varepsilon_k)e^{-\theta}S^-_jS^z_k\right) + \sinh^2(2\varepsilon_j)\left(e^{2\theta}S^+_jS^+_k + e^{-2\theta}S^-_jS^-_k\right),$$
$$T_3 = -4\cosh(2\varepsilon_j)\left(\sinh\xi\cosh\zeta\cosh\varepsilon_j + \cosh\xi\sinh\zeta\sinh\varepsilon_j\right)e^{\theta}S^+_j +$$
$$+\,2\sinh(2\varepsilon_j)\left(\sinh\xi\cosh\zeta\sinh\varepsilon_j + \cosh\xi\sinh\zeta\cosh\varepsilon_j\right)e^{\theta}S^+_j -$$
$$-\,2(t-s)\sinh(2\varepsilon_j)\left(\sinh\xi\cosh\zeta\cosh\varepsilon_j + \cosh\xi\sinh\zeta\sinh\varepsilon_j\right)e^{\theta}S^+_j +$$
$$+\,2\sinh(2\varepsilon_j)\sinh\xi\sinh\zeta\left((\gamma+\delta)\cosh\varepsilon_j + (\alpha+\beta)\sinh\varepsilon_j\right)e^{\theta}S^+_j +$$
$$+\,2\sinh(2\varepsilon_j)\cosh\xi\cosh\zeta\left((\alpha+\beta)\cosh\varepsilon_j + (\gamma+\delta)\sinh\varepsilon_j\right)e^{\theta}S^+_j -$$

$$
\begin{aligned}
&- 4\cosh(2\varepsilon_j)\left(\sinh\xi\cosh\zeta\cosh\varepsilon_j - \cosh\xi\sinh\zeta\sinh\varepsilon_j\right)e^{-\theta}S_j^- + \\
&+ 2\sinh(2\varepsilon_j)\left(\sinh\xi\cosh\zeta\sinh\varepsilon_j - \cosh\xi\sinh\zeta\cosh\varepsilon_j\right)e^{-\theta}S_j^- + \\
&+ 2(t-s)\sinh(2\varepsilon_j)\left(\sinh\xi\cosh\zeta\cosh\varepsilon_j - \cosh\xi\sinh\zeta\sinh\varepsilon_j\right)e^{-\theta}S_j^- + \\
&+ 2\sinh(2\varepsilon_j)\sinh\xi\sinh\zeta\left((\gamma+\delta)\cosh\varepsilon_j - (\alpha+\beta)\sinh\varepsilon_j\right)e^{-\theta}S_j^- + \\
&+ 2\sinh(2\varepsilon_j)\cosh\xi\cosh\zeta\left((\alpha+\beta)\cosh\varepsilon_j - (\gamma+\delta)\sinh\varepsilon_j\right)e^{-\theta}S_j^- + \\
&+ 2\sinh(2\xi)\sinh(2\zeta)S_j^z + 2(\gamma+\delta)\sinh(2\xi)\sinh(2\varepsilon_j)S_j^z - \\
&- 2(\alpha+\beta)\sinh(2\zeta)\sinh(2\varepsilon_j)S_j^z - 2(t-s)\sinh^2(2\varepsilon_j)S_j^z.
\end{aligned}
$$

Remark 5.3. *Let us check that the diagonal limit agrees with the expressions from Section 3.2.1. Multiplying each expression above by* $e^{2\zeta}$ *and considering* $\zeta \to -\infty$ *we obtain*

$$
\begin{aligned}
T_1 \to &\left(\cosh^2\varepsilon_j - \cosh^2\xi\right)\left(S_j^+S_k^- + S_j^-S_k^+ + 2\cosh(\varepsilon_j-\varepsilon_k)S_j^zS_k^z\right) = \\
&= \sinh(\varepsilon_j+\xi)\sinh(\varepsilon_j-\xi)\left(S_j^+S_k^- + S_j^-S_k^+ + 2\cosh(\varepsilon_j-\varepsilon_k)S_j^zS_k^z\right), \\
T_2 \to &-\left(\sinh\xi\cosh\varepsilon_j - \cosh\xi\sinh\varepsilon_j\right)^2 S_j^+S_k^- - \left(\sinh\xi\cosh\varepsilon_j + \cosh\xi\sinh\varepsilon_j\right)^2 S_j^-S_k^+ + \\
&\quad + 2\cosh(\varepsilon_j+\varepsilon_k)\left(\cosh^2\varepsilon_j - \cosh^2\xi\right)S_j^zS_k^z = \\
&= 2\cosh(\varepsilon_j+\varepsilon_k)\sinh(\varepsilon_j-\xi)\sinh(\varepsilon_j+\xi)S_j^zS_k^z - \\
&\quad - \sinh^2(\varepsilon_j-\xi)S_j^+S_k^- - \sinh^2(\varepsilon_j+\xi)S_j^-S_k^+, \\
T_3 \to &\left((\alpha+\beta)\sinh(2\varepsilon_j) - \sinh(2\xi)\right)S_j^z.
\end{aligned}
$$

This agrees, up to a scalar multiple, with (3.16).

5.5 The second family of conserved operators

Let us now consider the second family $\{\tilde{\tau}_j\}$ of the conserved operators obtained, as before, by (3.17) from the transfer matrix (5.2):

$$
\lim_{u\to-\varepsilon_j}(u+\varepsilon_j)t(u) = \eta^2\tilde{\tau}_j + \mathcal{O}(\eta^3).
$$

Proposition 5.4. *We can show that* $\tilde{\tau}_j = -\tau_j$*, thus, the second family is equivalent to the first one.*

Proof. Again, let us introduce $\vec{\varepsilon}$ into the notation, rewriting (5.7) as

$$\tau_j(\vec{\varepsilon}) = \sum_{k\neq j}^{\mathcal{L}} \frac{T_1(\vec{\varepsilon})}{\sinh(\varepsilon_j - \varepsilon_k)} + \sum_{k=1}^{\mathcal{L}} \frac{T_2(\vec{\varepsilon})}{\sinh(\varepsilon_j + \varepsilon_k)} + T_3(\vec{\varepsilon}).$$

Using the same argument as in Sections 3.2.2 and 4.2.2 it is easy to see that in this case

$$\tilde{\tau}_j(\vec{\varepsilon}) = \tau_j(-\vec{\varepsilon})^T\big|_{\theta_+\mapsto-\theta_+,\theta_-\mapsto-\theta_-} = \tau_j(-\vec{\varepsilon})^T\big|_{\theta\mapsto-\theta,t\mapsto-t,s\mapsto-s}.$$

Now consider the terms in (5.7) one by one. Start with the first term. It is straightforward to check that

$$T_1(-\vec{\varepsilon})^T = T_1(\vec{\varepsilon}) \quad \Rightarrow \quad \sum_{k\neq j}^{\mathcal{L}} \frac{T_1(-\vec{\varepsilon})^T}{\sinh(-\varepsilon_j + \varepsilon_k)} = -\sum_{k\neq j}^{\mathcal{L}} \frac{T_1(\vec{\varepsilon})}{\sinh(\varepsilon_j - \varepsilon_k)}.$$

For the second term let us consider separately the cases when $k \neq j$ and when $k = j$. It is easy to check that

$$\sum_{k\neq j}^{\mathcal{L}} \frac{T_2(-\vec{\varepsilon})^T\big|_{\theta\mapsto-\theta}}{\sinh(-\varepsilon_j - \varepsilon_k)} = -\sum_{k\neq j}^{\mathcal{L}} \frac{T_2(\vec{\varepsilon})}{\sinh(\varepsilon_j + \varepsilon_k)}.$$

Now consider $T_2(\vec{\varepsilon})$ with $k = j$. Using the properties of the spin-1/2 representation

$$S^+S^- = \frac{I}{2} + S^z,\ S^-S^+ = \frac{I}{2} - S^z,\ (S^z)^2 = \frac{I}{4},\ (S^+)^2 = (S^-)^2 = 0,$$
$$S^zS^+ = \frac{1}{2}S^+,\ S^+S^z = -\frac{1}{2}S^+,\ S^zS^- = -\frac{1}{2}S^-,\ S^-S^z = \frac{1}{2}S^-,$$

we obtain

$$\begin{aligned}
&T_2(\vec{\varepsilon}, k=j) = \\
&= -4\left(\sinh^2\xi\cosh^2\zeta\cosh^2\varepsilon_j + \cosh^2\xi\sinh^2\zeta\sinh^2\varepsilon_j\right) I - \\
&\quad - 2\sinh(2\xi)\sinh(2\zeta)\sinh(2\varepsilon_j)S_j^z - \\
&\quad - 2\cosh(2\varepsilon_j)\left(\sinh^2\xi\cosh^2\zeta\cosh^2\varepsilon_j - \cosh^2\xi\sinh^2\zeta\sinh^2\varepsilon_j\right) I - \\
&\quad - \frac{1}{2}\cosh(2\varepsilon_j)\sinh^2(2\varepsilon_j)I + \\
&\quad + 2\sinh(2\varepsilon_j)\left(\sinh\xi\cosh\zeta\cosh\varepsilon_j + \cosh\xi\sinh\zeta\sinh\varepsilon_j\right)\left(e^{-\theta}S_j^- + \cosh(2\varepsilon_j)e^{\theta}S_j^+\right) + \\
&\quad + 2\sinh(2\varepsilon_j)\left(\sinh\xi\cosh\zeta\cosh\varepsilon_j - \cosh\xi\sinh\zeta\sinh\varepsilon_j\right)\left(e^{\theta}S_j^+ + \cosh(2\varepsilon_j)e^{-\theta}S_j^-\right).
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
&T_2(-\vec{\varepsilon}, k=j)^T\big|_{\theta\mapsto-\theta} = \\
&= -4\left(\sinh^2\xi\cosh^2\zeta\cosh^2\varepsilon_j + \cosh^2\xi\sinh^2\zeta\sinh^2\varepsilon_j\right) I + \\
&\quad + 2\sinh(2\xi)\sinh(2\zeta)\sinh(2\varepsilon_j) S_j^z - \\
&\quad - 2\cosh(2\varepsilon_j)\left(\sinh^2\xi\cosh^2\zeta\cosh^2\varepsilon_j - \cosh^2\xi\sinh^2\zeta\sinh^2\varepsilon_j\right) I - \\
&\quad - \frac{1}{2}\cosh(2\varepsilon_j)\sinh^2(2\varepsilon_j) I + \\
&\quad - 2\sinh(2\varepsilon_j)\left(\sinh\xi\cosh\zeta\cosh\varepsilon_j - \cosh\xi\sinh\zeta\sinh\varepsilon_j\right)\left(e^{\theta}S_j^+ + \cosh(2\varepsilon_j)e^{-\theta}S_j^-\right) + \\
&\quad - 2\sinh(2\varepsilon_j)\left(\sinh\xi\cosh\zeta\cosh\varepsilon_j + \cosh\xi\sinh\zeta\sinh\varepsilon_j\right)\left(e^{-\theta}S_j^- + \cosh(2\varepsilon_j)e^{\theta}S_j^+\right).
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&\frac{1}{\sinh(2\varepsilon_j)}\left[T_2(\vec{\varepsilon}, k=j) - T_2(-\vec{\varepsilon}, k=j)^T\big|_{\theta\mapsto-\theta}\right] = \\
&= -4\sinh(2\xi)\sinh(2\zeta) S_j^z + \\
&\quad + 4\left(\sinh\xi\cosh\zeta\cosh\varepsilon_j + \cosh\xi\sinh\zeta\sinh\varepsilon_j\right)\left(e^{-\theta}S_j^- + \cosh(2\varepsilon_j)e^{\theta}S_j^+\right) + \\
&\quad + 4\left(\sinh\xi\cosh\zeta\cosh\varepsilon_j - \cosh\xi\sinh\zeta\sinh\varepsilon_j\right)\left(e^{\theta}S_j^+ + \cosh(2\varepsilon_j)e^{-\theta}S_j^-\right).
\end{aligned}
$$

Using $\cosh(2\varepsilon_j) + 1 = 2\cosh^2\varepsilon_j$ and $\cosh(2\varepsilon_j) - 1 = 2\sinh^2\varepsilon_j$ we can simplify

$$
\begin{aligned}
&\frac{1}{\sinh(2\varepsilon_j)}\left[T_2(\vec{\varepsilon}, k=j) - T_2(-\vec{\varepsilon}, k=j)^T\big|_{\theta\mapsto-\theta}\right] = \\
&= -4\sinh(2\xi)\sinh(2\zeta) S_j^z + \\
&\quad + 8\sinh\xi\cosh\zeta\cosh^3\varepsilon_j\left(e^{\theta}S_j^+ + e^{-\theta}S_j^-\right) + 8\cosh\xi\sinh\zeta\sinh^3\varepsilon_j\left(e^{\theta}S_j^+ - e^{-\theta}S_j^-\right).
\end{aligned}
$$

Now it remains to consider the last term in $\tau_j(\vec{\varepsilon}) + \tilde{\tau}_j(\vec{\varepsilon})$:

$$
\begin{aligned}
&T_3(\vec{\varepsilon_j}) + T_3(-\vec{\varepsilon_j})^T\big|_{\theta\mapsto-\theta, t\mapsto-t, s\mapsto-s} = \\
&= -8\cosh(2\varepsilon_j)\left(\sinh\xi\cosh\zeta\cosh\varepsilon_j + \cosh\xi\sinh\zeta\sinh\varepsilon_j\right) e^{\theta}S_j^+ - \\
&\quad - 8\cosh(2\varepsilon_j)\left(\sinh\xi\cosh\zeta\cosh\varepsilon_j - \cosh\xi\sinh\zeta\sinh\varepsilon_j\right) e^{-\theta}S_j^- + \\
&\quad + 4\sinh(2\varepsilon_j)\left(\sinh\xi\cosh\zeta\sinh\varepsilon_j + \cosh\xi\sinh\zeta\cosh\varepsilon_j\right) e^{\theta}S_j^+ + \\
&\quad + 4\sinh(2\varepsilon_j)\left(\sinh\xi\cosh\zeta\sinh\varepsilon_j - \cosh\xi\sinh\zeta\cosh\varepsilon_j\right) e^{-\theta}S_j^- + \\
&\quad + \sinh(2\xi)\sinh(2\zeta) S_j^z = \\
&= -8\cosh(2\varepsilon_j)\sinh\xi\cosh\zeta\cosh\varepsilon_j\left(e^{\theta}S_j^+ + e^{-\theta}S_j^-\right) - \\
&\quad - 8\cosh(2\varepsilon_j)\cosh\xi\sinh\zeta\sinh\varepsilon_j\left(e^{\theta}S_j^+ - e^{-\theta}S_j^-\right) +
\end{aligned}
$$

$$
\begin{aligned}
&+ 4\sinh(2\varepsilon_j)\sinh\xi\cosh\zeta\sinh\varepsilon_j\left(e^{\theta}S_j^+ + e^{-\theta}S_j^-\right) + \\
&+ 4\sinh(2\varepsilon_j)\cosh\xi\sinh\zeta\cosh\varepsilon_j\left(e^{\theta}S_j^+ - e^{-\theta}S_j^-\right) + \\
&+ 4\sinh(2\xi)\sinh(2\zeta)S_j^z = \\
= &-4\sinh\xi\cosh\zeta\left(2\cosh(2\varepsilon_j)\cosh\varepsilon_j - \sinh(2\varepsilon_j)\sinh\varepsilon_j\right)\left(e^{\theta}S_j^+ + e^{-\theta}S_j^-\right) - \\
&- 4\cosh\xi\sinh\zeta\left(2\cosh(2\varepsilon_j)\sinh\varepsilon_j - \sinh(2\varepsilon_j)\cosh\varepsilon_j\right)\left(e^{\theta}S_j^+ - e^{-\theta}S_j^-\right) + \\
&+ 4\sinh(2\xi)\sinh(2\zeta)S_j^z
\end{aligned}
$$

It is easy to see that

$$
\begin{aligned}
2\cosh(2\varepsilon_j)\cosh\varepsilon_j - \sinh(2\varepsilon_j)\sinh\varepsilon_j &= 2\cosh^3\varepsilon_j, \\
2\cosh(2\varepsilon_j)\sinh\varepsilon_j - \sinh(2\varepsilon_j)\cosh\varepsilon_j &= 2\sinh^3\varepsilon_j.
\end{aligned}
$$

Thus, we obtain

$$
T_3(\vec{\varepsilon_j}) + T_3(-\vec{\varepsilon_j})^T\big|_{\theta\mapsto-\theta,t\mapsto-t,s\mapsto-s} + \frac{1}{\sinh(2\varepsilon_j)}\left[T_2(\vec{\varepsilon}, k=j) - T_2(-\vec{\varepsilon}, k=j)^T\big|_{\theta\mapsto-\theta}\right] = 0,
$$

from which follows that $\tau_j(\vec{\varepsilon}) + \tilde{\tau}_j(\vec{\varepsilon}) = 0$. □

5.6 A special case

Note that our construction depends on 9 parameters in total coming from the expansion (5.5). We have freedom to adjust these parameters in order to obtain different constructions. In this section we consider one interesting special case when $\xi = \zeta = 0$. Then the conserved operators (5.7) reduce to the following (we have omitted the constant term for simplicity):

$$
\begin{aligned}
\tau_j = &\sum_{k\neq j}^{\mathcal{L}} \frac{\sinh^2(2\varepsilon_j)}{\sinh(\varepsilon_j - \varepsilon_k)}\left(S_j^+S_k^- + S_j^-S_k^+ + 2\cosh(\varepsilon_j - \varepsilon_k)S_j^zS_k^z\right) + \\
&+ \sum_{k\neq j}^{\mathcal{L}} \frac{\sinh^2(2\varepsilon_j)}{\sinh(\varepsilon_j + \varepsilon_k)}\left(e^{2\theta}S_j^+S_k^+ + e^{-2\theta}S_j^-S_k^- - 2\cosh(\varepsilon_j + \varepsilon_k)S_j^zS_k^z\right) + \\
&+ 2\sinh(2\varepsilon_j)\Big((\alpha+\beta)\cosh\varepsilon_j(e^{\theta}S_j^+ + e^{-\theta}S_j^-) + (\gamma+\delta)\sinh\varepsilon_j(e^{\theta}S_j^+ - e^{-\theta}S_j^-) - \\
&\qquad - (t-s)\sinh(2\varepsilon_j)S_j^z\Big).
\end{aligned}
$$

Define $\tau_j^* = \dfrac{\tau_j}{\sinh(2\varepsilon_j)}$. Then utilising

$$\coth(\varepsilon_j - \varepsilon_k) - \coth(\varepsilon_j + \varepsilon_k) = \frac{\sinh(2\varepsilon_k)}{\sinh(\varepsilon_j - \varepsilon_k)\sinh(\varepsilon_j + \varepsilon_k)}$$

we can rewrite

$$\begin{aligned}
\tau_j^* = &\sum_{k\neq j}^{\mathcal{L}} \frac{\sinh(2\varepsilon_j)}{\sinh(\varepsilon_j - \varepsilon_k)} \left(S_j^+ S_k^- + S_j^- S_k^+\right) + \\
&+ \sum_{k\neq j}^{\mathcal{L}} \frac{\sinh(2\varepsilon_j)}{\sinh(\varepsilon_j + \varepsilon_k)} \left(e^{2\theta} S_j^+ S_k^+ + e^{-2\theta} S_j^- S_k^-\right) + \\
&+ 2\sum_{k\neq j}^{\mathcal{L}} \frac{\sinh(2\varepsilon_j)\sinh(2\varepsilon_k)}{\sinh(\varepsilon_j - \varepsilon_k)\sinh(\varepsilon_j + \varepsilon_k)} S_j^z S_k^z + \\
&+ 2\Bigg((\alpha+\beta)\cosh\varepsilon_j(e^{\theta}S_j^+ + e^{-\theta}S_j^-) + (\gamma+\delta)\sinh\varepsilon_j(e^{\theta}S_j^+ - e^{-\theta}S_j^-) - \\
&\qquad - (t-s)\sinh(2\varepsilon_j)S_j^z\Bigg).
\end{aligned} \tag{5.8}$$

Let us look at this expression more closely. The terms with $S_j^+ S_k^- + S_j^- S_k^+$, $S_j^z S_k^z$ and S_j^z are analogous to the ones we had in the diagonal case (3.16). The linear terms with S_j^+ and S_j^- are analogous to the extra terms in (4.10). But now we also have a term $e^{2\theta} S_j^+ S_k^+ + e^{-2\theta} S_j^- S_k^-$, which is essentially different from all other terms. Thus, conserved operators (5.8) are neither equivalent to those obtained in the rational case (4.10) nor to those obtained in the diagonal case (3.16).

Remark 5.5. *Note that, since we have set $\zeta = 0$, we cannot take the diagonal limit anymore from the expression (5.8) (recall that the diagonal limit is taken by multiplying by $e^{2\zeta}$ and sending $\zeta \to -\infty$).*

Remark 5.6. *If we additionally set $\theta = s = t = 0$ the expression (5.8) will reduce to*

$$\begin{aligned}
\tau_j^* = &\sum_{k\neq j}^{\mathcal{L}} \frac{\sinh(2\varepsilon_j)}{\sinh(\varepsilon_j - \varepsilon_k)} \left(S_j^+ S_k^- + S_j^- S_k^+\right) + \sum_{k\neq j}^{\mathcal{L}} \frac{\sinh(2\varepsilon_j)}{\sinh(\varepsilon_j + \varepsilon_k)} \left(S_j^+ S_k^+ + S_j^- S_k^-\right) + \\
&+ 2\sum_{k\neq j}^{\mathcal{L}} \frac{\sinh(2\varepsilon_j)\sinh(2\varepsilon_k)}{\sinh(\varepsilon_j - \varepsilon_k)\sinh(\varepsilon_j + \varepsilon_k)} S_j^z S_k^z + \\
&+ 2\Big((\alpha+\beta)\cosh\varepsilon_j(S_j^+ + S_j^-) + (\gamma+\delta)\sinh\varepsilon_j(S_j^+ - S_j^-)\Big).
\end{aligned}$$

We notice that the structure is similar to that of the conserved operators for the elliptic

Gaudin system [ED15]

$$R_j = \sum_{k\neq j}^{\mathcal{L}} \left[\frac{k}{2}\,\mathrm{sn}(z_j - z_k)\left(S_j^+ S_k^+ + S_j^- S_k^-\right) + \frac{1}{\mathrm{sn}(z_j - z_k)}\left(S_j^+ S_k^- + S_j^- S_k^+\right) + \right.$$
$$\left. + \frac{\mathrm{cn}(z_j - z_k)\,\mathrm{dn}(z_j - z_k)}{\mathrm{sn}(z_j - z_k)} S_j^z S_k^z \right],$$

where $\mathrm{sn}(z),\ \mathrm{cn}(z),\ \mathrm{dn}(z)$ *are the doubly periodic elliptic Jacobi functions of modulus* k*. In particular, both expressions contain the term* $S_j^+ S_k^+ + S_j^- S_k^-$*.*

Now let us construct a Hamiltonian from these conserved operators, like we did for the rational case in Section 4.2.4. Consider

$$H \equiv \sum_{j=1}^{\mathcal{L}} \tau_j^* = \sum_{j=1}^{\mathcal{L}} \sum_{k\neq j}^{\mathcal{L}} \frac{\sinh(2\varepsilon_j)}{\sinh(\varepsilon_j - \varepsilon_k)}\left(S_j^+ S_k^- + S_j^- S_k^+\right) +$$
$$+ \sum_{j=1}^{\mathcal{L}} \sum_{k\neq j}^{\mathcal{L}} \frac{\sinh(2\varepsilon_j)}{\sinh(\varepsilon_j + \varepsilon_k)}\left(e^{2\theta} S_j^+ S_k^+ + e^{-2\theta} S_j^- S_k^-\right) +$$
$$+ 2\sum_{j=1}^{\mathcal{L}} \Bigg((\alpha+\beta)\cosh\varepsilon_j (e^{\theta} S_j^+ + e^{-\theta} S_j^-) + (\gamma+\delta)\sinh\varepsilon_j (e^{\theta} S_j^+ - e^{-\theta} S_j^-) -$$
$$- (t-s)\sinh(2\varepsilon_j) S_j^z \Bigg).$$

Writing it in the symmetric form we obtain

$$\begin{aligned} H = &\sum_{j=1}^{\mathcal{L}} \sum_{k=j+1}^{\mathcal{L}} \left(\frac{\sinh(2\varepsilon_j)}{\sinh(\varepsilon_j - \varepsilon_k)} - \frac{\sinh(2\varepsilon_k)}{\sinh(\varepsilon_j - \varepsilon_k)} \right)\left(S_j^+ S_k^- + S_j^- S_k^+\right) + \\ &+ \sum_{j=1}^{\mathcal{L}} \sum_{k=j+1}^{\mathcal{L}} \left(\frac{\sinh(2\varepsilon_j)}{\sinh(\varepsilon_j + \varepsilon_k)} + \frac{\sinh(2\varepsilon_k)}{\sinh(\varepsilon_j + \varepsilon_k)} \right)\left(e^{2\theta} S_j^+ S_k^+ + e^{-2\theta} S_j^- S_k^-\right) + \\ &+ 2(\alpha+\beta)\sum_{j=1}^{\mathcal{L}} \cosh\varepsilon_j (e^{\theta} S_j^+ + e^{-\theta} S_j^-) + \\ &+ 2(\gamma+\delta)\sum_{j=1}^{\mathcal{L}} \sinh\varepsilon_j (e^{\theta} S_j^+ - e^{-\theta} S_j^-) - 2(t-s)\sum_{j=1}^{\mathcal{L}} \sinh(2\varepsilon_j) S_j^z. \end{aligned} \tag{5.9}$$

This has a similar structure to the Hamiltonian (4.11), but contains the additional interaction terms $S_j^+ S_k^+$ and $S_j^- S_k^-$. These interactions terms are not so natural for interpretation as a fermionic model, as was described by (4.12). Nonetheless there may be other contexts for which (5.9) provides a potential physical model.

5.7 Summary

In this chapter we have investigated the open, trigonometric Richardson–Gaudin model from the BQISM with off-diagonal K-matrices. We have checked that the diagonal and rational limits agree with the constructions discussed in previous chapters. We have calculated the conserved operators in the quasi-classical limit (5.7) and proved that, like in all previous cases, the second family is equivalent to the first one. The difficulty in this case is that the expressions $T_1,\ T_2,\ T_3$ for the traces in (5.7) are quite cumbersome and hard to analyse. On the other hand, we have freedom of adjusting parameters to obtain different constructions as restrictions of the general one.

We have considered one special case obtained by setting $\xi = \zeta = 0$. This leads to the conserved operators (5.8). This expression is essentially different from the diagonal (3.16) and the rational (4.10) cases, which we considered previously. In fact, due to the way in which we restricted our parameters, it is no longer possible to take the diagonal limit of (5.8) (see Remark 5.5). In Remark 5.6 we notice certain similarity of (5.8) with the conserved operators for the elliptic Gaudin model. This suggests an equivalence between the trigonometric boundary construction and elliptic periodic construction, similar to the connection between the rational boundary construction and the trigonometric twisted-periodic construction, which we established previously. Further investigation is required to explore this connection. Finally, we have constructed a Hamiltonian as a linear combination of the conserved operators.

Chapter 6

Some open questions about the bosonic Lax operator

So far in this thesis we have studied models based on the spin-1/2 Lax operator, trigonometric (2.17) and rational (2.18). In this Chapter we venture into new territory and consider the bosonic Lax operator (6.2) below, which can be applied to a range of physical models, including the two-site Bose–Hubbard model for quantum tunneling [ZLMG03, LH06, LFTS06]. First of all, in Section 6.1 we attempt to include the boundary into the quantum tunneling model by applying the BQISM construction introduced in Section 2.4. It turns out that this does not increase the number of independent conserved operators. Expanding the transfer matrix in powers of the spectral parameter u yields only one non-trivial conserved operator.

Next, in Section 6.2 we turn to the case of the q-deformed bosonic Lax operator ((6.9) below). Note that for the trigonometric spin-1/2 Lax operator (2.17) the rational and the quasi-classical[1] limits are well-defined and commute. On the one hand, the quasi-classical limit of the rational limit (2.18) gives

$$\ell_{aj}(u) = \begin{pmatrix} S_j^z & S_j^- \\ S_j^+ & -S_j^z \end{pmatrix}.$$

On the other hand, first taking the quasi-classical limit (2.24)

$$\begin{pmatrix} S_j^z \cosh u & S_j^- \\ S_j^+ & -S_j^z \cosh u \end{pmatrix},$$

[1]By the quasi-classical limit we mean, as before, the first non-trivial coefficient in the expansion in η as $\eta \to 0$.

and then the rational limit gives the same result. The situation turns out to be different for the q-deformed bosonic Lax operator (6.9). Neither the rational nor the quasi-classical limit is well-defined in this case. We can modify the Lax operator to make them well-defined, but the limits do not commute. Next, in Section 6.3 we look at the rational and quasi-classical limits for the BAE and we confirm that these limits do not commute. Finally, in Section 6.4 we try to overcome this issue by considering a specific monodromy matrix (6.27) instead of the Lax operator. Both rational and the quasi-classical limits are defined for this monodromy matrix, but other technical issues arise. We discuss several directions for future research involving the bosonic Lax operator.

6.1 Quantum tunneling model

6.1.1 Periodic case

First of all let us review the quantum tunneling model (without boundary) [LH06, LFTS06]. It can be described by the two-site Bose–Hubbard Hamiltonian

$$H = \frac{k}{8}(N_1 - N_2)^2 - \frac{\mu}{2}(N_1 - N_2) - \frac{\mathcal{E}}{2}(b_1^\dagger b_2 + b_1 b_2^\dagger). \tag{6.1}$$

where $b_j^\dagger$ and b_j $(j = 1, 2)$ are the bosonic creation and annihilation operators satisfying

$$[b_j, b_k^\dagger] = \delta_{jk} I, \quad [b_j, b_k] = [b_j^\dagger, b_k^\dagger] = 0,$$

and $N_j = b_j^\dagger b_j$ are the corresponding number operators, $k, \mathcal{E} \in \mathbb{C}$ are the coupling constants.

In order to merge this model into QISM formalism, we start with the rational R-matrix (2.3) and consider the following Lax operator:

$$L(u) = \begin{pmatrix} (1 + \eta u)I + \eta^2 N & \eta b \\ \eta b^\dagger & I \end{pmatrix}. \tag{6.2}$$

It is an operator in $\mathrm{End}(V \otimes W)$, where $V = \mathbb{C}^2$, as before, and W is a representation space for the bosonic algebra (note that it has to be infinite-dimensional). It is easy to check that the RLL relation (2.6) is satisfied.

Consider the monodromy matrix (2.7) with $\gamma = 0$ and $\mathcal{L} = 2$

$$T(u) = L_1(u - \varepsilon_1)L_2(u - \varepsilon_2).$$

Note that we are free to shift the spectral parameter in the Lax operator without the loss of generality. Thus, we can make it more symmetric by setting $\varepsilon_1 = \eta^{-1} - \omega,\ \ \varepsilon_2 = \eta^{-1} + \omega$, where $\omega \in \mathbb{C}$. It is also convenient to rescale the monodromy matrix by η^{-2}. Finally, the monodromy matrix for the quantum tunneling model is given by

$$T(u) = \eta^{-2}L_1(u - \eta^{-1} + \omega)L_2(u - \eta^{-1} - \omega). \tag{6.3}$$

It is an operator in $\mathrm{End}(V \otimes W_1 \otimes W_2)$, where W_1 and W_2 are two representation spaces of the bosonic algebra. One can write (6.3) as an operator valued 2×2-matrix in the auxiliary space V:

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

with the entries

$$\begin{aligned}
A(u) &= (u^2 - \omega^2)I + \eta u N + \eta^2 N_1 N_2 - \eta\omega(N_1 - N_2) + b_1 b_2^\dagger, \\
B(u) &= (u + \omega + \eta N_1)b_2 + \eta^{-1} b_1, \\
C(u) &= b_1^\dagger(u - \omega + \eta N_2) + \eta^{-1} b_2^\dagger, \\
D(u) &= b_1^\dagger b_2 + \eta^{-2} I.
\end{aligned}$$

Thus, the transfer matrix $t(u) = A(u) + D(u)$ is given by

$$t(u) = \eta^2 N_1 N_2 - \eta\omega(N_1 - N_2) + b_1 b_2^\dagger + b_1^\dagger b_2 + (u^2 - \omega^2 + \eta^{-2})I. \tag{6.4}$$

The Hamiltonian (6.1) for the quantum tunneling model is constructed as follows from the transfer matrix (6.4):

$$H = -\frac{\mathcal{E}}{2}\left[t(u) + (\omega^2 - \eta^{-2} - u^2)I - \eta u N - \frac{\eta^2}{4}N^2\right].$$

6.1.2 Boundary case

Now let us include the boundary based on the BQISM construction described in Section 2.4. The rational R-matrix (2.3) satisfies the reflection equations (2.32) with the rational diagonal K-matrices given by

$$K^-(u) = \begin{pmatrix} \xi^- + u & 0 \\ 0 & \xi^- - u \end{pmatrix}, \quad K^+(u) = \begin{pmatrix} \xi^+ + u + \eta & 0 \\ 0 & \xi^+ - u - \eta \end{pmatrix}. \tag{6.5}$$

Remark 6.1. *These are obtained as a rational limit from the trigonometric diagonal matrices (2.40) and (2.41). Note that here we do not perform the shift $u \mapsto u - \eta/2$ as we did when we considered the spin-$1/2$ Lax operator.*

Let us consider the following double-row monodromy matrix, acting in $V \otimes W_1 \otimes W_2$:

$$\begin{aligned} T(u) = \eta^{-2} L_1(u - \eta^{-1} + \omega) L_2(u - \eta^{-1} - \omega) K^-(u) \times \\ \times (L_2(-u - \eta^{-1} - \omega))^{-1} (L_1(-u - \eta^{-1} + \omega))^{-1}, \end{aligned}$$

and the corresponding transfer matrix

$$t(u) = \mathrm{tr}(K^+(u) T(u)), \tag{6.6}$$

where the trace is taken in the auxiliary space V.

We would like to write it out explicitly, similarly to (6.4). This requires more effort than in the periodic case. First of all, let us calculate the inverse $(L(u))^{-1}$:

$$(L(u))^{-1} = \frac{1}{1 + \eta u - \eta^2} \begin{pmatrix} I & -\eta b \\ -\eta b^\dagger & (1 + \eta u - \eta^2) I + \eta^2 N \end{pmatrix}.$$

Thus, the factor in the transfer matrix (6.6) coming from the inverse Lax operators is

$$(1 + \eta(-u - \eta^{-1} - \omega) - \eta^2)(1 + \eta(-u - \eta^{-1} + \omega) - \eta^2) = \eta^2((u + \eta)^2 - \omega^2).$$

It is convenient to rescale the transfer matrix (6.6) as follows:

$$t(u) \mapsto ((u + \eta)^2 - \omega^2) t(u).$$

Denote

$$\begin{pmatrix} A_1(u) & B_1(u) \\ C_1(u) & D_1(u) \end{pmatrix} = \eta^{-2} L_1(u - \eta^{-1} + \omega) L_2(u - \eta^{-1} - \omega),$$

$$\begin{pmatrix} A_2(u) & B_2(u) \\ C_2(u) & D_2(u) \end{pmatrix} = ((u + \eta)^2 - \omega^2)(L_2)^{-1}(-u - \eta^{-1} - \omega)(L_1)^{-1}(-u - \eta^{-1} + \omega) =$$

$$= \eta^{-2} \begin{pmatrix} I & -\eta b_1 \\ -\eta b_1^\dagger & (-\eta u - \eta\omega - \eta^2)I + \eta^2 N_1 \end{pmatrix} \times$$
$$\times \begin{pmatrix} I & -\eta b_2 \\ -\eta b_2^\dagger & (\eta u - \eta\omega - \eta^2)I + \eta^2 N_2 \end{pmatrix},$$

so that

$$t(u) = \operatorname{tr}\left(K^+(u) \begin{pmatrix} A_1(u) & B_1(u) \\ C_1(u) & D_1(u) \end{pmatrix} K^-(u) \begin{pmatrix} A_2(u) & B_2(u) \\ C_2(u) & D_2(u) \end{pmatrix} \right),$$

or, substituting the K-matrices (6.5),

$$\begin{aligned} t(u) = {} & (\xi^+ + u + \eta)(\xi^- + u)A_1(u)A_2(u) + (\xi^+ + u + \eta)(\xi^- - u)B_1(u)C_2(u) + \\ & + (\xi^+ - u - \eta)(\xi^- + u)C_1(u)B_2(u) + (\xi^+ - u - \eta)(\xi^- - u)D_1(u)D_2(u). \end{aligned} \tag{6.7}$$

From Section 6.1.1 we have

$$\begin{aligned} A_1(u) &= (u^2 - \omega^2)I + \eta u N + \eta^2 N_1 N_2 - \eta\omega(N_1 - N_2) + b_1 b_2^\dagger, \\ B_1(u) &= (u + \omega + \eta N_1)b_2 + \eta^{-1} b_1, \\ C_1(u) &= b_1^\dagger(u - \omega + \eta N_2) + \eta^{-1} b_2^\dagger, \\ D_1(u) &= b_1^\dagger b_2 + \eta^{-2} I. \end{aligned}$$

Analogously, we calculate

$$\begin{aligned} A_2(u) &= b_1^\dagger b_2 + \eta^{-2} I, \\ B_2(u) &= (u + \eta - \omega - \eta N_1)b_2 - \eta^{-1} b_1, \\ C_2(u) &= b_1^\dagger(u + \eta + \omega - \eta N_2) - \eta^{-1} b_2^\dagger, \\ D_2(u) &= ((u + \eta)^2 - \omega^2)I - \eta(u + \eta)N + \eta^2 N_1 N_2 - \eta\omega(N_1 - N_2) + b_1 b_2^\dagger. \end{aligned}$$

Thus, one can see that (6.7) is a 4th order polynomial in u. Let us calculate it explicitly. We will use the bosonic commutation relations

$$b^\dagger b = N, \quad bb^\dagger = N + I, \quad b^\dagger N = (N - I)b^\dagger, \quad bN = (N + I)b.$$

Let start by expanding the terms

$$\begin{aligned} A_1(u)A_2(u) = {} & \Big[(u^2 - \omega^2)I + \eta u N + \eta^2 N_1 N_2 - \eta\omega(N_1 - N_2) + b_1 b_2^\dagger\Big]\Big[b_1^\dagger b_2 + \eta^{-2} I\Big] = \\ & = u^2\left(b_1^\dagger b_2 + \eta^{-2} I\right) + u\left(\eta N b_1^\dagger b_2 + \eta^{-1} N\right) - \omega^2 b_1^\dagger b_2 - \omega^2 \eta^{-2} I + \\ & + \eta^2 N_1 N_2 b_1^\dagger b_2 - \eta\omega(N_1 - N_2) b_1^\dagger b_2 + 2N_1 N_2 + N_2 - \eta^{-1}\omega(N_1 - N_2) + \end{aligned}$$

$$
\begin{aligned}
&+\eta^{-2}b_1b_2^\dagger,\\
B_1(u)C_2(u) &= \Big[(u+\omega+\eta N_1)b_2+\eta^{-1}b_1\Big]\Big[b_1^\dagger(u+\eta+\omega-\eta N_2)-\eta^{-1}b_2^\dagger\Big]=\\
&= u^2b_1^\dagger b_2+u\Big(2\omega b_1^\dagger b_2+\eta(N_1-N_2)b_1^\dagger b_2+\eta^{-1}(N_1-N_2)\Big)+\omega^2b_1^\dagger b_2+\\
&+\eta\omega(N_1-N_2)b_1^\dagger b_2+\eta^{-1}\omega(N_1-N_2)-\eta^2N_1N_2b_1^\dagger b_2-2N_1N_2-N_2+\\
&+I-\eta^{-2}b_1b_2^\dagger,\\
C_1(u)B_2(u) &= \Big[b_1^\dagger(u-\omega+\eta N_2)+\eta^{-1}b_2^\dagger\Big]\Big[(u+\eta-\omega-\eta N_1)b_2-\eta^{-1}b_1\Big]=\\
&= u^2b_1^\dagger b_2+u\Big(2(\eta-\omega)b_1^\dagger b_2-\eta(N_1-N_2)b_1^\dagger b_2-\eta^{-1}(N_1-N_2)\Big)+\\
&+(\omega^2-2\omega\eta)b_1^\dagger b_2+\eta\omega(N_1-N_2)b_1^\dagger b_2+\eta^{-1}\omega(N_1-N_2)-\eta^2N_1N_2b_1^\dagger b_2+\\
&+2\eta^2N_2b_1^\dagger b_2-2N_1N_2+N_2-\eta^{-2}b_1b_2^\dagger,\\
D_1(u)D_2(u) &= \Big[b_1^\dagger b_2+\eta^{-2}I\Big]\times\\
&\times\Big[((u+\eta)^2-\omega^2)I-\eta(u+\eta)N+\eta^2N_1N_2-\eta\omega(N_1-N_2)+b_1b_2^\dagger\Big]=\\
&= u^2\Big(b_1^\dagger b_2+\eta^{-2}I\Big)+u\Big(2\eta b_1^\dagger b_2+2\eta^{-1}I-\eta Nb_1^\dagger b_2-\eta^{-1}N\Big)-\\
&-(\omega^2-2\eta\omega)b_1^\dagger b_2-\eta^2Nb_1^\dagger b_2-\eta^{-2}\omega^2I+I+2N_1N_2-N_2+\eta^2N_1N_2b_1^\dagger b_2+\\
&+(\eta^2-\eta\omega)(N_1-N_2)b_1^\dagger b_2-\eta^{-1}\omega(N_1-N_2)+\eta^{-2}b_1b_2^\dagger.
\end{aligned}
$$

Then,

$$
\begin{aligned}
&(\xi^++u+\eta)(\xi^-+u)A_1(u)A_2(u)=\\
&=u^4\Big[b_1^\dagger b_2+\eta^{-2}I\Big]+\\
&\quad+u^3\Big[\eta Nb_1^\dagger b_2+\eta^{-1}N+(\xi^++\xi^-)b_1^\dagger b_2+(\xi^++\xi^-)\eta^{-2}I+\eta b_1^\dagger b_2+\eta^{-1}I\Big]+\\
&\quad+u^2\Big[(\xi^++\xi^-)\eta Nb_1^\dagger b_2+(\xi^++\xi^-)\eta^{-1}N+\eta^2Nb_1^\dagger b_2+N-\omega^2b_1^\dagger b_2-\omega^2\eta^{-2}I+\\
&\qquad+\eta^2N_1N_2b_1^\dagger b_2-\eta\omega(N_1-N_2)b_1^\dagger b_2+2N_1N_2+N_2-\eta^{-1}\omega(N_1-N_2)+\eta^{-2}b_1b_2^\dagger+\\
&\qquad+\xi^+\xi^-b_1^\dagger b_2+\xi^-\eta b_1^\dagger b_2+\xi^+\xi^-\eta^{-2}I+\xi^-\eta^{-1}I\Big]+\\
&\quad+u\Big[(\xi^++\xi^-+\eta)(-\omega^2b_1^\dagger b_2-\omega^2\eta^{-2}I+\eta^2N_1N_2b_1^\dagger b_2-\eta\omega(N_1-N_2)b_1^\dagger b_2+\\
&\qquad+2N_1N_2+N_2-\eta^{-1}\omega(N_1-N_2)+\eta^{-2}b_1b_2^\dagger)+\xi^+\xi^-\eta Nb_1^\dagger b_2+\xi^-\eta^2Nb_1^\dagger b_2+\\
&\qquad+\xi^+\xi^-\eta^{-1}N+\xi^-N\Big]+\\
&\quad+\Big[(\xi^+\xi^-+\eta\xi^-)(-\omega^2b_1^\dagger b_2-\omega^2\eta^{-2}I+\eta^2N_1N_2b_1^\dagger b_2-\eta\omega(N_1-N_2)b_1^\dagger b_2+2N_1N_2+\\
&\qquad+N_2-\eta^{-1}\omega(N_1-N_2)+\eta^{-2}b_1b_2^\dagger)\Big],
\end{aligned}
$$

$$
\begin{aligned}
&(\xi^+ + u + \eta)(\xi^- - u)B_1(u)C_2(u) = \\
&= u^4\Big[-b_1^\dagger b_2\Big] + \\
&\quad + u^3\Big[-2\omega b_1^\dagger b_2 - \eta(N_1-N_2)b_1^\dagger b_2 - \eta^{-1}(N_1-N_2) + (\xi^- - \xi^+)b_1^\dagger b_2 - \eta b_1^\dagger b_2\Big] + \\
&\quad + u^2\Big[2(\xi^- - \xi^+)\omega b_1^\dagger b_2 - 2\eta\omega b_1^\dagger b_2 + (\xi^- - \xi^+)\eta(N_1-N_2)b_1^\dagger b_2 - \eta^2(N_1-N_2)b_1^\dagger b_2 + \\
&\qquad\quad + (\xi^- - \xi^+)\eta^{-1}(N_1-N_2) - (N_1-N_2) - \omega^2 b_1^\dagger b_2 - \eta\omega(N_1-N_2)b_1^\dagger b_2 - \\
&\qquad\quad - \eta^{-1}\omega(N_1-N_2) + \eta^2 N_1N_2 b_1^\dagger b_2 + 2N_1N_2 + N_2 - I + \eta^{-2}b_1b_2^\dagger + \xi^+\xi^- b_1^\dagger b_2 + \\
&\qquad\quad + \eta\xi^- b_1^\dagger b_2\Big] + \\
&\quad + u\Big[(\xi^- - \xi^+ - \eta)(\omega^2 b_1^\dagger b_2 + \eta\omega(N_1-N_2)b_1^\dagger b_2 + \eta^{-1}\omega(N_1-N_2) - \eta^2 N_1N_2 b_1^\dagger b_2 - \\
&\qquad - 2N_1N_2 - N_2 + I - \eta^{-2}b_1b_2^\dagger) + 2\xi^+\xi^-\omega b_1^\dagger b_2 + 2\xi^-\eta\omega b_1^\dagger b_2 + \xi^+\xi^-\eta(N_1-N_2)b_1^\dagger b_2 + \\
&\qquad + \eta^2\xi^-(N_1-N_2)b_1^\dagger b_2 + \xi^+\xi^-\eta^{-1}(N_1-N_2) + \xi^-(N_1-N_2)\Big] + \\
&\quad + \Big[(\xi^+\xi^- + \eta\xi^-)(\omega^2 b_1^\dagger b_2 + \eta\omega(N_1-N_2)b_1^\dagger b_2 + \eta^{-1}\omega(N_1-N_2) - \eta^2 N_1N_2 b_1^\dagger b_2 - \\
&\qquad - 2N_1N_2 - N_2 + I - \eta^{-2}b_1b_2^\dagger)\Big],
\end{aligned}
$$

$$
\begin{aligned}
&(\xi^+ - u - \eta)(\xi^- + u)C_1(u)B_2(u) = \\
&= u^4\Big[-b_1^\dagger b_2\Big] + \\
&\quad + u^3\Big[-2(\eta-\omega)b_1^\dagger b_2 + \eta(N_1-N_2)b_1^\dagger b_2 + \eta^{-1}(N_1-N_2) + (\xi^+ - \xi^-)b_1^\dagger b_2 - \eta b_1^\dagger b_2\Big] + \\
&\quad + u^2\Big[2(\xi^+ - \xi^-)(\eta-\omega)b_1^\dagger b_2 - 2\eta(\eta-\omega)b_1^\dagger b_2 - (\xi^+ - \xi^-)\eta(N_1-N_2)b_1^\dagger b_2 + \\
&\qquad\quad + \eta^2(N_1-N_2)b_1^\dagger b_2 - (\xi^+ - \xi^-)\eta^{-1}(N_1-N_2) + (N_1-N_2) - (\omega^2 - 2\omega\eta)b_1^\dagger b_2 - \\
&\qquad\quad - \eta\omega(N_1-N_2)b_1^\dagger b_2 - \eta^{-1}\omega(N_1-N_2) + \eta^2 N_1N_2 b_1^\dagger b_2 - 2\eta^2 N_2 b_1^\dagger b_2 + 2N_1N_2 - \\
&\qquad\quad - N_2 + \eta^{-2}b_1b_2^\dagger + \xi^+\xi^- b_1^\dagger b_2 - \eta\xi^- b_1^\dagger b_2\Big] + \\
&\quad + u\Big[(\xi^+ - \xi^- - \eta)((\omega^2 - 2\omega\eta)b_1^\dagger b_2 + \eta\omega(N_1-N_2)b_1^\dagger b_2 + \eta^{-1}\omega(N_1-N_2) - \\
&\qquad - \eta^2 N_1N_2 b_1^\dagger b_2 + 2\eta^2 N_2 b_1^\dagger b_2 - 2N_1N_2 + N_2 - \eta^{-2}b_1b_2^\dagger) + 2\xi^+\xi^-(\eta-\omega)b_1^\dagger b_2 - \\
&\qquad - 2\eta\xi^-(\eta-\omega)b_1^\dagger b_2 - \xi^+\xi^-\eta(N_1-N_2)b_1^\dagger b_2 + \eta^2\xi^-(N_1-N_2)b_1^\dagger b_2 - \\
&\qquad - \xi^+\xi^-\eta^{-1}(N_1-N_2) + \xi^-(N_1-N_2)\Big] + \\
&\quad + \Big[(\xi^+\xi^- - \eta\xi^-)((\omega^2 - 2\omega\eta)b_1^\dagger b_2 + \eta\omega(N_1-N_2)b_1^\dagger b_2 + \eta^{-1}\omega(N_1-N_2) - \\
&\qquad - \eta^2 N_1N_2 b_1^\dagger b_2 + 2\eta^2 N_2 b_1^\dagger b_2 - 2N_1N_2 + N_2 - \eta^{-2}b_1b_2^\dagger)\Big],
\end{aligned}
$$

$$
\begin{aligned}
&(\xi^+ - u - \eta)(\xi^- - u)D_1(u)D_2(u) = \\
&= u^4\Big[b_1^\dagger b_2 + \eta^{-2}I\Big] + \\
&\quad + u^3\Big[2\eta b_1^\dagger b_2 + 2\eta^{-1}I - \eta N b_1^\dagger b_2 - \eta^{-1}N - (\xi^+ + \xi^-)b_1^\dagger b_2 + \eta b_1^\dagger b_2 - \\
&\qquad\quad - (\xi^+ + \xi^-)\eta^{-2} + \eta^{-1}I\Big] + \\
&\quad + u^2\Big[-2(\xi^+ + \xi^-)\eta b_1^\dagger b_2 + 2\eta^2 b_1^\dagger b_2 - 2(\xi^+ + \xi^-)\eta^{-1} + 2I + (\xi^+ + \xi^-)\eta N b_1^\dagger b_2 - \\
&\qquad\quad - \eta^2 N b_1^\dagger b_2 + (\xi^+ + \xi^-)\eta^{-1}N - N - (\omega^2 - 2\eta\omega)b_1^\dagger b_2 - \eta^2 N b_1^\dagger b_2 - \eta^{-2}\omega^2 I + I + \\
&\qquad\quad + 2N_1N_2 - N_2 + \eta^2 N_1 N_2 b_1^\dagger b_2 + (\eta^2 - \eta\omega)(N_1 - N_2)b_1^\dagger b_2 - \eta^{-1}\omega(N_1 - N_2) + \\
&\qquad\quad + \eta^{-2}b_1 b_2^\dagger + \xi^+\xi^- b_1^\dagger b_2 - \eta\xi^- b_1^\dagger b_2 + \xi^+\xi^-\eta^{-2}I - \eta^{-1}\xi^-\Big] + \\
&\quad + u\Big[-(\xi^+ + \xi^- - \eta)(-(\omega^2 - 2\eta\omega)b_1^\dagger b_2 - \eta^2 N b_1^\dagger b_2 - \eta^{-2}\omega^2 I + I + 2N_1N_2 - N_2 + \\
&\qquad + \eta^2 N_1 N_2 b_1^\dagger b_2 + (\eta^2 - \eta\omega)(N_1 - N_2)b_1^\dagger b_2 - \eta^{-1}\omega(N_1 - N_2) + \eta^{-2}b_1 b_2^\dagger) + \\
&\qquad + 2\eta\xi^+\xi^- b_1^\dagger b_2 - 2\eta^2\xi^- b_1^\dagger b_2 + 2\eta^{-1}\xi^+\xi^- - 2\xi^- - \eta\xi^+\xi^- N b_1^\dagger b_2 + \eta^2\xi^- N b_1^\dagger b_2 - \\
&\qquad - \eta^{-1}\xi^+\xi^- N + \xi^- N\Big] + \\
&\quad + \Big[(\xi^+\xi^- - \eta\xi^-)(-(\omega^2 - 2\eta\omega)b_1^\dagger b_2 - \eta^2 N b_1^\dagger b_2 - \eta^{-2}\omega^2 I + I + 2N_1N_2 - N_2 + \\
&\qquad + \eta^2 N_1 N_2 b_1^\dagger b_2 + (\eta^2 - \eta\omega)(N_1 - N_2)b_1^\dagger b_2 - \eta^{-1}\omega(N_1 - N_2) + \eta^{-2}b_1 b_2^\dagger)\Big].
\end{aligned}
$$

Substituting these into (6.7) we obtain the final result

$$
t(u) = 2\eta^{-2}Iu^4 + 4\eta^{-1}Iu^3 + Fu^2 + \eta(F - 2I)u + 2\xi^+\xi^-(1 - \omega^2\eta^{-2})I,
$$

where

$$
\begin{aligned}
F = \eta^2&\Big[4(N_1 - 1)N_2 b_1^\dagger b_2\Big] + \\
&+ \eta\Big[-4\omega(N_1 - N_2)b_1^\dagger b_2 + 4\xi^- N_1 b_1^\dagger b_2 + 4\xi^+ N_2 b_1^\dagger b_2 + \\
&\qquad + 4\omega b_1^\dagger b_2 - 4\xi^- b_1^\dagger b_2\Big] + \\
&+ \Big[-4\omega^2 b_1^\dagger b_2 + 4\omega(\xi^- - \xi^+)b_1^\dagger b_2 + 4\xi^+\xi^- b_1^\dagger b_2 + 8N_1N_2 + 2I\Big] + \\
&+ \eta^{-1}\Big[-4\omega(N_1 - N_2) + 4\xi^- N_1 + 4\xi^+ N_2 - 2(\xi^+ + \xi^-)\Big] + \\
&+ \eta^{-2}\Big[4b_1 b_2^\dagger + 2\xi^+\xi^- - 2\omega^2\Big].
\end{aligned}
$$

Thus, we come to the conclusion that, although by including the boundary we have doubled the degree of the transfer matrix as a polynomial in u, we still obtain only one non-trivial conserved operator F. So, as in the case of Richardson–Gaudin models (see Propositions 3.2, 4.4 and 5.4), including the boundary does not lead to increasing the number of independent conserved quantities. Since the bosonic Lax operator (6.2) has not fulfilled our expectations of obtaining something more interesting in the boundary case, we will now look at the q-deformed bosonic Lax operator for the rest of this chapter.

6.2 The q-deformed bosonic Lax operator

In this section we will investigate the q-deformed verision of the bosonic model described in Section 6.1.1. Let us start with the trigonometric R-matrix (2.2). Denoting

$$\lambda = e^u, \; q = e^{\eta/2}$$

we can rewrite it as

$$R(\lambda) = \frac{1}{\lambda q^2 - \lambda^{-1} q^{-2}} \begin{pmatrix} \lambda q^2 - \lambda^{-1} q^{-2} & 0 & 0 & 0 \\ 0 & \lambda - \lambda^{-1} & q^2 - q^{-2} & 0 \\ 0 & q^2 - q^{-2} & \lambda - \lambda^{-1} & 0 \\ 0 & 0 & 0 & \lambda q^2 - \lambda^{-1} q^{-2} \end{pmatrix}. \tag{6.8}$$

The q-deformed bosonic Lax operator was originally introduced in [Kun07a]. We will use a slightly modified form, as in [DILZ11]:

$$L(\lambda) = \begin{pmatrix} \lambda q^{2N+1} - \lambda^{-1} q^{-2N-1} & (q^4 - q^{-4})^{1/2} b_q \\ (q^4 - q^{-4})^{1/2} b_q^\dagger & \lambda q^{-2N-1} + \lambda^{-1} q^{2N+1} \end{pmatrix}, \tag{6.9}$$

where the q-boson operators b_q, $b_q^\dagger$ and N satisfy ([Mac89, Bie89])

$$[b_q, b_q^\dagger] = \frac{q^{2(2N+1)} + q^{-2(2N+1)}}{q^2 + q^{-2}}, \; [b_q, N] = b_q, \; [b_q^\dagger, N] = -b_q^\dagger.$$

Note that when $q \to 1$, b_q and $b_q^\dagger$ become the usual bosonic destruction and creation operators b and $b^\dagger$ satisfying $[b, b^\dagger] = I$.

The Lax operator given by (6.9) is an operator in $\mathrm{End}(V \otimes W)$, where W is a representation space for the q-boson algebra (infinite-dimensional). One can check that (6.9) satisfies the RLL relation (2.6) in $\mathrm{End}(V \otimes V \otimes W)$ together with the R-matrix (6.8), which in the new variables takes the form

$$R_{12}(\lambda/\mu)L_1(\lambda)L_2(\mu) = L_2(\mu)L_1(\lambda)R_{12}(\lambda/\mu). \tag{6.10}$$

The problem with the Lax operator (6.9) is that we cannot directly take the rational or the quasi-classical limit from it, as we did for the spin-1/2 Lax operator (2.17).

- To be able to take the *rational limit* we need the Lax operator to satisfy

$$L(\lambda)\big|_{q=1,\lambda=1} = 0, \tag{6.11}$$

  but for (6.9) we have $L(\lambda)\big|_{q=1,\lambda=1} = \mathrm{diag}(0, 2)$.

- To be able to take the *quasi-classical limit* we need

$$L(\lambda)\big|_{q=1} \propto I, \tag{6.12}$$

  but here we have $L(\lambda)\big|_{q=1} = \mathrm{diag}(\lambda - \lambda^{-1}, \lambda + \lambda^{-1})$.

The good news is that we can modify the Lax operator (6.9) without violating (6.10) to make conditions (6.11), (6.12) satisfied. Obviously, we can rescale it and make a change of variables, but we can also make a following transform:

$$L(\lambda) \mapsto AL(\lambda)B, \tag{6.13}$$

where $A, B \in \mathrm{End}(V)$ satisfy

$$\begin{aligned} R_{12}(\lambda)A_1A_2 &= A_2A_1R_{12}(\lambda), \\ R_{12}(\lambda)B_1B_2 &= B_2B_1R_{12}(\lambda). \end{aligned} \tag{6.14}$$

Remark 6.2. *In the following we will only consider A and B diagonal, in which case the conditions (6.14) are automatically satisfied for the solution (6.8). For general values of the deformation parameter q, diagonal solutions of (6.14) are the most general. However, for the rational solution (2.3), all $A, B \in$ End(V) satisfy (6.14).*

6.2.1 Rational limit

Consider the transform (6.13) with $A = B = \operatorname{diag}\big(1, (q^4 - q^{-4})^{1/2}\big)$:

$$L(\lambda) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & (q^4 - q^{-4})^{1/2} \end{pmatrix} L(\lambda) \begin{pmatrix} 1 & 0 \\ 0 & (q^4 - q^{-4})^{1/2} \end{pmatrix}.$$

The modified Lax operator, which still satisfies (6.10), is now of the form

$$L(\lambda)' = \begin{pmatrix} \lambda q^{2N+1} - \lambda^{-1} q^{-2N-1} & (q^4 - q^{-4}) b_q \\ (q^4 - q^{-4}) b_q^\dagger & (q^4 - q^{-4})(\lambda q^{-2N-1} + \lambda^{-1} q^{2N+1}) \end{pmatrix}. \tag{6.15}$$

Note that now the condition (6.11) for taking the rational limit is satisfied. To take the rational limit we introduce an additional parameter ν:

$$\lambda = e^u \mapsto e^{\nu u}, \quad q = e^{\eta/2} \mapsto e^{\nu\eta/2},$$

divide (6.15) by ν and consider the expansion as $\nu \to 0$. We have

$$e^{\nu u} = 1 + \nu u + \mathcal{O}(\nu^2), \quad e^{\nu\eta/2} = 1 + \nu\eta/2 + \mathcal{O}(\nu^2),$$

and, hence,

$$\begin{aligned} q^4 - q^{-4} &= 4\eta\nu + \mathcal{O}(\nu^2), \\ \lambda q^{2N+1} - \lambda^{-1} q^{-2N-1} &= 2u\nu + \eta(2N+1)\nu + \mathcal{O}(\nu^2), \\ (q^4 - q^{-4})(\lambda q^{-2N-1} + \lambda^{-1} q^{2N+1}) &= 8\eta\nu + \mathcal{O}(\nu^2). \end{aligned}$$

Thus, the rational limit of (6.15) gives

$$L(u)' = \begin{pmatrix} 2u + \eta(2N+1) & 4\eta b \\ 4\eta b^\dagger & 8\eta \end{pmatrix}. \tag{6.16}$$

Note that we cannot directly take the quasi-classical limit of (6.16), because $L(u)'|_{\eta=0}$ is not proportional to I, but we can easily fix it by the following transformation[2]:

$$L(u)' \mapsto \frac{1}{4} \begin{pmatrix} (2\eta)^{1/2} & 0 \\ 0 & (2\eta)^{-1/2} \end{pmatrix} L(u)' \begin{pmatrix} (2\eta)^{1/2} & 0 \\ 0 & (2\eta)^{-1/2} \end{pmatrix},$$

[2] Note that it is a twist, so it will lead to a scaling factor in the BAE, which we will discuss in Section 6.3.

together with a variable change $u \mapsto u - \eta/2 + \eta^{-1}$. This turns (6.16) into the rational bosonic Lax operator (6.2)

$$L(u) = \begin{pmatrix} (1+\eta u)I + \eta^2 N & \eta b \\ \eta b^\dagger & I \end{pmatrix},$$

and the quasi-classical limit of (6.2) gives

$$\ell(u) = \begin{pmatrix} u & b \\ b^\dagger & 0 \end{pmatrix}. \tag{6.17}$$

Thus, we worked out how to take first the rational limit and then the quasi-classical limit of (6.9). Now let us try to take the limits in the different order.

6.2.2 Quasi-classical limit

For taking the quasi-classical limit we modify the the Lax operator (6.9) as follows. Consider the mapping

$$L(\lambda) \mapsto (q^4 - q^{-4})^{1/2} L(\lambda)$$

together with the change of variable $(q^4 - q^{-4})^{1/2}\lambda \mapsto \lambda$. The modified Lax operator, which still satisfies (2.6), is now of the form

$$L(\lambda)'' = \begin{pmatrix} \lambda q^{2N+1} - (q^4 - q^{-4})\lambda^{-1} q^{-2N-1} & (q^4 - q^{-4}) b_q \\ (q^4 - q^{-4}) b_q^\dagger & \lambda q^{-2N-1} + (q^4 - q^{-4})\lambda^{-1} q^{2N+1} \end{pmatrix}. \tag{6.18}$$

The condition (6.12) for taking the quasi-classical limit is now satisfied. As $\eta \to 0$, we have $q = e^{\eta/2} = 1 + \mathcal{O}(\eta^2)$ and $q^4 - q^{-4} = e^{2\eta} - e^{-2\eta} = 4\eta + \mathcal{O}(\eta^2)$. Thus, expanding (6.18) in η we obtain

$$L(\lambda)'' = \lambda I + \eta \begin{pmatrix} \lambda(N+1/2) - 4\lambda^{-1} & 4b \\ 4b^\dagger & -(\lambda(N+1/2) - 4\lambda^{-1}) \end{pmatrix} + \mathcal{O}(\eta^2).$$

Let us make a change of variables $\lambda \mapsto 2\sqrt{2}\lambda$. Then we obtain

$$L(\lambda)'' = 2\sqrt{2}\lambda I + 2\sqrt{2}\eta \begin{pmatrix} \lambda N + 1/2(\lambda - \lambda^{-1}) & \sqrt{2} b \\ \sqrt{2} b^\dagger & -(\lambda N + 1/2(\lambda - \lambda^{-1})) \end{pmatrix} + \mathcal{O}(\eta^2).$$

The quasi-classical limit gives

$$\ell(\lambda) = \frac{1}{\lambda}\begin{pmatrix} \lambda N + 1/2(\lambda - \lambda^{-1}) & \sqrt{2}b \\ \sqrt{2}b^\dagger & -(\lambda N + 1/2(\lambda - \lambda^{-1})) \end{pmatrix}. \tag{6.19}$$

It has no obvious rational limit, because

$$\ell(\lambda)|_{\lambda=1} = \begin{pmatrix} N & \sqrt{2}b \\ \sqrt{2}b^\dagger & -N \end{pmatrix} \neq 0,$$

and it cannot be made so by a simple shift of (6.19) by a diagonal matrix.

Thus, the rational and quasi-classical limits do not commute for the Lax operator (6.9). There is still a possibility that we just have not found a suitable transformation. To check this, let us have a look at the BAE. The BAE do not depend on the basis, unlike the Lax operator, so we are only allowed rescalings and variable changes.

6.3 Rational and quasi-classical limits of the Bethe Ansatz Equations

Let us consider the BAE to check whether the rational and quasi-classical limits commute there. In order to derive the BAE we apply the algebraic Bethe Ansatz for the twisted-periodic QISM construction described in Section 2.2.

6.3.1 Rational limit

Starting with the Lax operator (6.15) we construct the monodromy matrix as follows:

$$T(\lambda)' = L_1(\lambda/\alpha_1)' L_2(\lambda/\alpha_2)'.$$

It is an operator in $\mathrm{End}(V \otimes W_1 \otimes W_2)$. As usual, let us write it as a 2×2-matrix in the auxiliary space V:

$$T(\lambda)' = \begin{pmatrix} A(\lambda)' & B(\lambda)' \\ C(\lambda)' & D(\lambda)' \end{pmatrix}.$$

Let us calculate the diagonal entries of the monodromy matrix:

$$A(\lambda)' = \left(\frac{\lambda}{\alpha_1}q^{2N_1+1} - \frac{\alpha_1}{\lambda}q^{-2N_1-1}\right)\left(\frac{\lambda}{\alpha_2}q^{2N_2+1} - \frac{\alpha_2}{\lambda}q^{-2N_2-1}\right) + (q^4-q^{-4})^2 b_{q1}b^\dagger_{q2},$$
$$D(u)' = (q^4-q^{-4})^2 b^\dagger_{q1}b_{q2} + \\ + (q^4-q^{-4})^2\left(\frac{\lambda}{\alpha_1}q^{-2N_1-1} + \frac{\alpha_1}{\lambda}q^{2N_1+1}\right)\left(\frac{\lambda}{\alpha_2}q^{-2N_2-1} + \frac{\alpha_2}{\lambda}q^{2N_2+1}\right).$$

In order to facilitate taking the rational limit let us rewrite these in terms of variables u and η. Recall that $\lambda = e^u,\ q = e^{\eta/2}$ and assume $\alpha_1 = e^{\varepsilon_1},\ \alpha_2 = e^{\varepsilon_2}$. Then we have

$$A(u)' = 4\sinh\big(u-\varepsilon_1+\eta(N_1+1/2)\big)\sinh\big(u-\varepsilon_2+\eta(N_2+1/2)\big) + (2\sinh(2\eta))^2 b_{q1}b^\dagger_{q2},$$
$$D(u)' = (2\sinh(2\eta))^2 b^\dagger_{q1}b_{q2} + \\ + 4(2\sinh(2\eta))^2\cosh\big(u-\varepsilon_1-\eta(N_1+1/2)\big)\cosh\big(u-\varepsilon_2-\eta(N_2+1/2)\big).$$

Consider the action of $A(u)'$ and $D(u)'$ on the vacuum state $|0\rangle\otimes|0\rangle$ (the state $|0\rangle$ in this case is determined by the conditions $N|0\rangle = b|0\rangle = 0$):

$$A(u)'|0\rangle\otimes|0\rangle = a(u)'|0\rangle\otimes|0\rangle,\ D(u)'|0\rangle\otimes|0\rangle = d(u)'|0\rangle\otimes|0\rangle,$$

where

$$a(u)' = 4\sinh(u-\varepsilon_1+\eta/2)\sinh(u-\varepsilon_2+\eta/2),$$
$$d(u)' = 4(2\sinh(2\eta))^2\cosh(u-\varepsilon_1-\eta/2)\cosh(u-\varepsilon_2-\eta/2).$$

Thus, the BAE are given by (cf. (2.15))

$$\frac{\sinh(v_k-\varepsilon_1+\eta/2)\sinh(v_k-\varepsilon_2+\eta/2)}{(2\sinh(2\eta))^2\cosh(v_k-\varepsilon_1-\eta/2)\cosh(v_k-\varepsilon_2-\eta/2)} = \prod_{j\neq k}^{N}\frac{\sinh(v_k-v_j-\eta)}{\sinh(v_k-v_j+\eta)}. \tag{6.20}$$

Now, in the rational limit from (6.20) we obtain

$$\frac{(v_k-\varepsilon_1+\eta/2)(v_k-\varepsilon_2+\eta/2)}{16\eta^2} = \prod_{j\neq k}^{N}\frac{v_k-v_j-\eta}{v_k-v_j+\eta}. \tag{6.21}$$

Let us also make a change of variables $v_k \mapsto v_k - \eta/2 + \eta^{-1}$ in (6.21), as we did for the Lax operator in Section 6.2.1. Finally, we obtain the following BAE:

$$\frac{(1+\eta(v_k-\varepsilon_1))(1+\eta(v_k-\varepsilon_2))}{16\eta^2} = \prod_{j\neq k}^{N}\frac{v_k-v_j-\eta}{v_k-v_j+\eta}. \tag{6.22}$$

These agree with the BAE for the non-deformed bosonic Lax operator $L(u)$ (6.2) and the twisted monodromy matrix $T(u) = ML_1(u-\varepsilon_1)L_2(u-\varepsilon_2)$[3], where $M = \mathrm{diag}((4\eta)^{-1}, 4\eta)$. Note that we cannot directly take the quasi-classical limit of (6.22), but if we consider the monodromy matrix without the twist $T(u) = L_1(u-\varepsilon_1)L_2(u-\varepsilon_2)$, then the BAE are

$$(1+\eta(v_k-\varepsilon_1))(1+\eta(v_k-\varepsilon_2)) = \prod_{j\neq k}^{N} \frac{v_k - v_j - \eta}{v_k - v_j + \eta}. \tag{6.23}$$

and the quasi-classical limit of the BAE (6.23) is given by

$$2v_k - \varepsilon_1 - \varepsilon_2 = -2\sum_{j\neq k}^{N} \frac{1}{v_k - v_j}. \tag{6.24}$$

6.3.2 Quasi-classical limit

Now let us first consider the quasi-classical limit and then the rational limit. Start with the Lax operator (6.18) and construct the monodromy matrix as

$$T(\lambda)'' = L_1(\lambda/\alpha_1)'' L_2(\lambda/\alpha_2)'' = \begin{pmatrix} A(\lambda)'' & B(\lambda)'' \\ C(\lambda)'' & D(\lambda)'' \end{pmatrix}.$$

We have

$$\begin{aligned} A(\lambda)'' &= \frac{\lambda^2}{\alpha_1\alpha_2} q^{2(N_1+N_2+1)} - (q^4 - q^{-4})\left(\frac{\alpha_2}{\alpha_1} q^{2(N_1-N_2)} + \frac{\alpha_1}{\alpha_2} q^{2(N_2-N_1)}\right) + \\ &+ (q^4-q^{-4})^2 \frac{\alpha_1\alpha_2}{\lambda^2} q^{-2(N_1+N_2+1)} + (q^4-q^{-4})^2 b_{q1} b_{q2}^\dagger, \\ D(\lambda)'' &= \frac{\lambda^2}{\alpha_1\alpha_2} q^{-2(N_1+N_2+1)} + (q^4 - q^{-4})\left(\frac{\alpha_1}{\alpha_2} q^{2(N_1-N_2)} + \frac{\alpha_2}{\alpha_1} q^{2(N_2-N_1)}\right) + \\ &+ (q^4-q^{-4})^2 \frac{\alpha_1\alpha_2}{\lambda^2} q^{2(N_1+N_2+1)} + (q^4-q^{-4})^2 b_{q1}^\dagger b_{q2}, \end{aligned}$$

and the action on the vacuum state $|0\rangle \otimes |0\rangle$ gives

$$\begin{aligned} a(\lambda)'' &= \frac{\lambda^2}{\alpha_1\alpha_2} q^2 - (q^4-q^{-4})\left(\frac{\alpha_1}{\alpha_2} + \frac{\alpha_2}{\alpha_1}\right) + (q^4-q^{-4})^2 \frac{\alpha_1\alpha_2}{\lambda^2} q^{-2}, \\ d(\lambda)'' &= \frac{\lambda^2}{\alpha_1\alpha_2} q^{-2} + (q^4-q^{-4})\left(\frac{\alpha_1}{\alpha_2} + \frac{\alpha_2}{\alpha_1}\right) + (q^4-q^{-4})^2 \frac{\alpha_1\alpha_2}{\lambda^2} q^{2}. \end{aligned}$$

[3]We include this twist to be consistent with the twist we performed on the Lax operator in Section 6.2.1.

Taking the quasi-classical expansion (using $q^4 - q^{-4} = 4\eta + \mathcal{O}(\eta^2)$) we obtain

$$a(\lambda)'' = \frac{\lambda^2}{\alpha_1\alpha_2} + \eta\left(\frac{\lambda^2}{\alpha_1\alpha_2} - 4\frac{\alpha_1^2+\alpha_2^2}{\alpha_1\alpha_2}\right) + \mathcal{O}(\eta^2),$$
$$d(\lambda)'' = \frac{\lambda^2}{\alpha_1\alpha_2} - \eta\left(\frac{\lambda^2}{\alpha_1\alpha_2} - 4\frac{\alpha_1^2+\alpha_2^2}{\alpha_1\alpha_2}\right) + \mathcal{O}(\eta^2).$$

The BAE are given by

$$\frac{a(\lambda)''}{d(\lambda)''} = \prod_{j\neq k}^{N} \frac{\sinh(v_k - v_j - \eta)}{\sinh(v_k - v_j + \eta)}. \tag{6.25}$$

The left hand side of (6.25) in the quasi-classical expansion gives

$$\frac{a(\lambda)''}{d(\lambda)''} = \frac{\lambda^2 + \eta\big(\lambda^2 - 4(\alpha_1^2+\alpha_2^2)\big)}{\lambda^2 - \eta\big(\lambda^2 - 4(\alpha_1^2+\alpha_2^2)\big)} = 1 + 2\eta\left(1 - 4\lambda^{-2}(\alpha_1^2+\alpha_2^2)\right) + \mathcal{O}(\eta^2).$$

In terms of variables $u,\ \varepsilon_1,\ \varepsilon_2$ we have

$$4\lambda^{-2}\left(\alpha_1^2+\alpha_2^2\right) = 4e^{-2u}\left(e^{2\varepsilon_1} + e^{2\varepsilon_2}\right).$$

The quasi-classical expansion of right hand side of (6.25) is, as before (cf. 2.3.1),

$$\prod_{j\neq k}^{N} \frac{\sinh(v_k - v_j - \eta)}{\sinh(v_k - v_j + \eta)} = 1 - 2\eta \sum_{i\neq k}^{N} \coth(v_k - v_i) + \mathcal{O}(\eta^2).$$

Thus, the quasi-classical expansion of the BAE (6.25) gives

$$1 - 4e^{-2v_k}\left(e^{2\varepsilon_1} + e^{2\varepsilon_2}\right) = -\sum_{j\neq k}^{N} \coth(v_k - v_j). \tag{6.26}$$

Let us try to take the rational limit from here. In order to eliminate the constant term in the rational limit of the left hand side consider a change of variables $v_k \mapsto v_k + \ln\sqrt{8}$. Then

$$1 - 4e^{-2v_k}\left(e^{2\varepsilon_1} + e^{2\varepsilon_2}\right) \mapsto 1 - \frac{1}{2}e^{-2v_k}\left(e^{2\varepsilon_1} + e^{2\varepsilon_2}\right).$$

Now introduce the rational parameter ν and consider the Taylor expansion as $\nu \to 0$:

$$1 - \frac{1}{2}e^{-2\nu v_k}\left(e^{2\nu\varepsilon_1} + e^{2\nu\varepsilon_2}\right) = 1 - \frac{1}{2}\big(1 - 2\nu v_k + \mathcal{O}(\nu^2)\big)\big(2 + 2\nu\varepsilon_1 + 2\nu\varepsilon_2 + \mathcal{O}(\nu^2)\big) =$$
$$= \nu(2v_k - \varepsilon_1 - \varepsilon_2) + \mathcal{O}(\nu^2).$$

On the right hand side we have

$$\coth\big(\nu(v_k - v_j)\big) = \frac{1}{\nu(v_k - v_j)} + \frac{\nu(v_k - v_j)}{3} + \mathcal{O}(\nu^2).$$

Thus, in the rational limit from (6.26) we obtain

$$0 = \sum_{j\neq k}^{N} \frac{1}{v_k - v_j},$$

which does not agree with (6.24). In fact, it is easy to see that these equations have no solution. We come to a conclusion that the rational and quasi-classical limits do not commute on the level of the BAE.

6.4 An alternative Lax operator

In this section we consider an alternative approach to overcome technical difficulties in taking the rational and quasi-classical limits, which we have encountered in this chapter so far. The idea consists in taking a special form of the monodromy matrix instead of the original Lax operator (6.9). This monodromy matrix is defined as follows

$$T(\lambda) = L_1(\lambda)L_2(i\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}, \tag{6.27}$$

where $L(\lambda)$ is the q-deformed Lax operator (6.9). Let us calculate its entries:

$$\begin{aligned}
A(\lambda) &= \left(\lambda q^{2N_1+1} - \lambda^{-1}q^{-2N_1-1}\right)\left(i\lambda q^{2N_2+1} + i\lambda^{-1}q^{-2N_2-1}\right) + (q^4 - q^{-4})b_{q1}b_{q2}^\dagger = \\
&= i\left(\lambda^2 q^{2(N_1+N_2+1)} - \lambda^{-2}q^{-2(N_1+N_2+1)} + q^{2(N_1-N_2)} - q^{-2(N_1-N_2)}\right) + (q^4 - q^{-4})b_{q1}b_{q2}^\dagger = \\
&= 2i\Big[\sinh\big(2u + \eta(N_1 + N_2 + 1)\big) + \sinh\big(\eta(N_1 - N_2)\big)\Big] + 2\sinh(2\eta)b_{q1}b_{q2}^\dagger, \\
B(\lambda) &= \left(\lambda q^{2N_1+1} - \lambda^{-1}q^{-2N_1-1}\right)(q^4 - q^{-4})^{1/2}b_{q2} + \\
&\quad + (q^4 - q^{-4})^{1/2}b_{q1}\left(i\lambda q^{-2N_2-1} - i\lambda^{-1}q^{2N_2+1}\right) = \\
&= 2\sqrt{2}(\sinh(2\eta))^{1/2}\Big[\sinh(u + \eta(N_1 + 1/2))b_{q2} + i\sinh(u - \eta(N_2 + 1/2))b_{q1}\Big], \\
C(\lambda) &= (q^4 - q^{-4})^{1/2}b_{q1}^\dagger\left(i\lambda q^{2N_2+1} + i\lambda^{-1}q^{-2N_2-1}\right) + \\
&\quad + \left(\lambda q^{-2N_1-1} + \lambda^{-1}q^{2N_1+1}\right)(q^4 - q^{-4})^{1/2}b_{q2}^\dagger = \\
&= 2\sqrt{2}(\sinh(2\eta))^{1/2}\Big[\cosh(u - \eta(N_1 + 1/2))b_{q2}^\dagger + i\cosh(u + \eta(N_2 + 1/2))b_{q1}^\dagger\Big],
\end{aligned}$$

$$\begin{aligned}D(\lambda) &= (q^4 - q^{-4})b_{q1}^\dagger b_{q2} + \left(\lambda q^{-2N_1-1} + \lambda^{-1} q^{2N_1+1}\right)\left(i\lambda q^{-2N_2-1} - i\lambda^{-1} q^{2N_2+1}\right) = \\ &= i\left(\lambda^2 q^{-2(N_1+N_2+1)} - \lambda^{-2} q^{2(N_1+N_2+1)} + q^{2(N_1-N_2)} - q^{-2(N_1-N_2)}\right) + (q^4 - q^{-4})b_{q1}^\dagger b_{q2} = \\ &= 2i\Big[\sinh\big(2u - \eta(N_1+N_2+1)\big) + \sinh\big(\eta(N_1-N_2)\big)\Big] + 2\sinh(2\eta) b_{q1}^\dagger b_{q2}.\end{aligned}$$

Note that, unlike the Lax operator (6.9), the monodromy matrix (6.27) satisfies both conditions (6.11) and (6.12) required for taking the rational and quasi-classical limits. Let us see what we obtain in these limits.

6.4.1 Rational limit

In the rational limit we obtain

$$\begin{aligned}A^{rat}(u) &= 4iu + 4\eta\big(i(N_1+1/2) + b_1 b_2^\dagger\big), \\ B^{rat}(u) &= 4\eta^{1/2} u\big(ib_1 + b_2\big) + 4\eta^{3/2}\big((N_1+1/2)b_2 - ib_1(N_2+1/2)\big), \\ C^{rat}(u) &= 4\eta^{1/2}\big(ib_1^\dagger + b_2^\dagger\big), \\ D^{rat}(u) &= 4iu - 4\eta\big(i(N_2+1/2) - b_1^\dagger b_2\big).\end{aligned}$$

We encounter the following technical difficulty in taking the quasi-classical limit of these expressions: in the expressions above, in particular in $B^{rat}(u)$ and $C^{rat}(u)$, we have $\eta^{1/2}$ as a leading power, but in the quasi-classical expansion of the R-matrix (2.3) η is the leading power.

6.4.2 Quasi-classical limit

If we try to take the quasi-classical limit directly from (6.27) above we encounter the same problem. The expansions of $B(\lambda)$ and $C(\lambda)$ have $\eta^{1/2}$ as the leading power:

$$\begin{aligned}A(\lambda) &= i\Big[\lambda^2\big(1 + \eta(N_1+N_2+1)\big) - \lambda^{-2}\big(1 - \eta(N_1+N_2+1)\big) + \\ &\qquad + \big(1 + \eta(N_1-N_2)\big) - \big(1 - \eta(N_1-N_2)\big)\Big] + 4\eta b_1 b_2^\dagger + \mathcal{O}(\eta^2) = \\ &= i(\lambda^2 - \lambda^{-2}) \times \\ &\quad \times \left[1 + \frac{\eta}{\lambda^2 - \lambda^{-2}}\Big((\lambda^2 + \lambda^{-2})(N_1+N_2+1) + 2(N_1-N_2) - 4ib_1 b_2^\dagger\Big) + \mathcal{O}(\eta^2)\right], \\ B(\lambda) &= 2\eta^{1/2}\Big[\lambda\big(1 + \eta(N_1+1/2)\big) - \lambda^{-1}\big(1 - \eta(N_1+1/2)\big) + \mathcal{O}(\eta^2)\Big] b_2 +\end{aligned}$$

$$
\begin{aligned}
&+ 2\eta^{1/2} i\Big[\lambda\big(1-\eta(N_2+1/2)\big) - \lambda^{-1}\big(1+\eta(N_2+1/2)\big) + \mathcal{O}(\eta^2)\Big] b_1 = \\
&= 2\eta^{1/2}\Big[(\lambda-\lambda^{-1})(ib_1+b_2) + \eta(\lambda+\lambda^{-1})\left((N_1+1/2)b_2 - ib_1(N_2+1/2)\right) + \mathcal{O}(\eta^2)\Big], \\
C(\lambda) &= 2\eta^{1/2} b_1^\dagger i\Big[\lambda\big(1+\eta(N_2+1/2)\big) + \lambda^{-1}\big(1-\eta(N_2+1/2)\big) + \mathcal{O}(\eta^2)\Big] + \\
&+ 2\eta^{1/2}\Big[\lambda\big(1-\eta(N_1+1/2)\big) + \lambda^{-1}\big(1+\eta(N_1+1/2)\big) + \mathcal{O}(\eta^2)\Big] b_2^\dagger = \\
&= 2\eta^{1/2}\Big[(\lambda+\lambda^{-1})(ib_1^\dagger+b_2^\dagger) + \eta(\lambda-\lambda^{-1})\Big(ib_1^\dagger(N_2+1/2) - (N_1+1/2)b_2^\dagger\Big) + \mathcal{O}(\eta^2)\Big], \\
D(\lambda) &= i\Big[\lambda^2\big(1-\eta(N_1+N_2+1)\big) - \lambda^{-2}\big(1+\eta(N_1+N_2+1)\big) + \\
&\quad + \big(1+\eta(N_1-N_2)\big) - \big(1-\eta(N_1-N_2)\big)\Big] + 4\eta b_1^\dagger b_2 + \mathcal{O}(\eta^2) = \\
&= i(\lambda^2-\lambda^{-2}) \times \\
&\quad \times \left[1 + \frac{\eta}{\lambda^2-\lambda^{-2}}\Big(-(\lambda^2+\lambda^{-2})(N_1+N_2+1) + 2(N_1-N_2) - 4ib_1^\dagger b_2\Big) + \mathcal{O}(\eta^2)\right].
\end{aligned}
$$

Further investigation is needed to better understand this problem.

Chapter 7

Conclusions

7.1 Summary

The *leitmotif* of this thesis is: What does it mean for a quantum integrable model to have a "boundary"? Sklyanin developed the BQISM method to include open spin chains into the QISM formalism. While for spin chains there is an obvious geometric interpretation of the boundary, for other models the situation is not as clear. We mainly focused on answering this question for Richardson–Gaudin models obtained in the quasi-classical limit from Sklyanin's BQISM construction[1]. We have investigated these models systematically and explored the connections between them. Below we summarise the main results that we have obtained.

- First of all, in Section 2.5 we introduced a generalised version of Sklyanin's BQISM construction, which depends on a complex parameter ρ. This extends an approach presented in [KZ94] and allows to interpolate between the boundary and twisted-periodic constructions. Sklyanin's boundary construction is obtained by setting $\rho = 0$ and the twisted-periodic construction is obtained in the limit as $\rho \to \infty$. We refer to this limit as the attenuated limit. The attenuated limits of various constructions are considered in Sections 2.5.1, 2.5.2 and 4.1.
- In Chapter 3 we investigated Richardson–Gaudin models obtained from the BQISM with diagonal K-matrices, assuming the quasi-classical expansion of the boundary parameters. We explored connections between the boundary and the twisted-periodic constructions, both trigonometric and rational. We showed that the rational boundary construction is equivalent to the trigonometric twisted-periodic

[1] We also briefly consider the two-site Bose–Hubbard model with "boundary" in Section 6.1.2

construction. Also, the trigonometric boundary construction is equivalent to its rational limit. We demonstrated these equivalences on the level of the BAE, conserved operators and their eigenvalues. Thus, we come to the conclusion that including diagonal boundary terms does not extend the class of Richardson–Gaudin models beyond results obtained from the twisted-periodic construction. The connections are summarised in Figure 3.2.

- Next, we considered the quasi-classical limit of the BQISM construction with off-diagonal K-matrices. We started with the rational case in Chapter 4. Unlike the diagonal case, including off-diagonal boundary terms does lead to a new model. In particular, by considering a linear combination of the conserved operators, we were able to construct an integrable generalisation of the $p + ip$ Hamiltonian with extra terms (4.12), which can be interpreted as an interaction of the system with its environment. Thus, we have a simple physical interpretation of the "boundary" in this case. The external interaction terms break the $\mathfrak{u}(1)$ symmetry of the model and, thus, the algebraic Bethe Ansatz is not obviously applicable. We applied the recently developed off-diagonal Bethe Ansatz method [WYCS15] to calculate the exact energy spectrum and derived the BAE whose roots parametrise it. Recently Claeys et al [CBN16] presented the explicit wave function and computed exact correlation functions for this model.

- Finally, in Chapter 5 we considered the trigonometric off-diagonal case and calculated conserved operators obtained in the quasi-classical limit from this construction. The expression for the conserved operators involves 9 free parameters, which gives us freedom to adjust these parameters in order to construct different integrable models. We considered one special case and discussed why conserved operators in this case are in general not equivalent to any of those considered in previous chapters. In fact, they have a similar form to conserved operators of the elliptic Gaudin model [ED15], which suggests a possible connection with the elliptic case. Exploring this connection requires further investigation. We have also constructed a Hamiltonian in this special case, which has a similar form as the Hamiltonian (4.11) from Chapter 4, but contains some extra interaction terms. We leave the interpretation of these terms for future work.

7.2 Future work

There are several directions for future research to continue the work started in this thesis, which are summarised below.

- We observed that in all the above cases including a "boundary" does not increase the number of independent conserved operators (although it doubles the degree of the transfer matrix as the polynomial in u). In the case of Richardson–Gaudin models the two families of conserved operators obtained in the quasi-classical limit turn out to be equivalent (see Propositions 3.2, 4.4 and 5.4). The same happens in the case of the two-site Bose–Hubbard model (see Section 6.1.2). In the future, we would like to explore this pattern further and check whether it is true in general.
- After studying the rational and trigonometric BQISM constructions the next natural step is to consider the elliptic BQISM construction based on the elliptic R-matrix. It would be interesting to investigate the elliptic Richardson–Gaudin model obtained in the quasi-classical limit from the elliptic BQISM construction and establish its trigonometric limit. Also, we would like to explore a possible connection between the trigonometric boundary construction and elliptic periodic construction identified in Remark 5.6, similar to the connection between the rational boundary construction and the trigonometric twisted-periodic construction, which we have established previously.
- We have observed that some results from this thesis hold not only for spin-1/2, but for higher spins (see Remark 4.5). It would be interesting to systematically generalise our results to the higher spin case. Another possible generalisation goes in the direction of considering higher rank algebras $\mathfrak{su}(n)$ instead of $\mathfrak{su}(2)$.
- In Chapter 6 we discussed several technical difficulties concerning the q-deformed bosonic Lax operator. This opens various avenues for future research. In particular, we would like to further investigate implications of the fact that the rational and quasi-classical limits do not commute in this case. Furthermore, we would like to apply the methods developed in this thesis to study spin-boson models and, in particular, try to approach problems related to integrability of the Rabi model [Bra11, BZ15].

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