Fundamental R-matrix for the quantum integrable model with symmetry algebra $GL_q(2)$

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1 Introduction and overview of the main results

As we know from quantum mechanics, every quantum system can be described by a Hilbert space $\mathcal{H}$, an algebra of observables $\mathcal{A}$ and a Hamiltonian H, which is some special observable defining the dynamics of the system. Any other observable A changes according to the following well-known formula (look [3]):

$$\frac{dA}{dt} = [H, A].$$

Observable is called the quantum *integral of motion* (IM), if it commutes with the Hamiltonian, i.e. is invariant under change of time. Quantum system is called *integrable*, if there is a complete set of mutually commuting IM. The term "complete" is not quiet clear, especially in infinite case, but from the beginning it can be understood in the following way: the set of IM is *complete*, if they are independent, but any other IM, commuting with all of them, can be written as a function of them (is not independent). The main problem is to construct these mutually commuting IM for a given quantum system. It was solved by the *Quantum Inverse Scattering Method* (QISM, look for example [4]). The main idea of this method is briefly described below.

Some of the real 1+1-dimensional quantum field systems can be approximated by the discrete ("lattice") quantum models, which are easier to solve. So, it is reasonable to restrict ourselves on *lattice models*. The most well-known examples are quantum spin models of the magnetic chains. The field-theoretical models will appear as their continuous space limits.

Homogenious 1+1-dimensional lattice model can be described by the representation of its *symmetry algebra*. In this case

$$\mathcal{H} = \bigotimes_{n=1}^{N} h_n = h_1 \otimes h_2 \otimes ... \otimes h_n \otimes ... \otimes h_N,$$

$$\mathcal{A} = \bigotimes_{n=1}^{N} \mathcal{U}_n = \mathcal{U}_1 \otimes \mathcal{U}_2 \otimes ... \otimes \mathcal{U}_n \otimes ... \otimes \mathcal{U}_N,$$

where each $\mathcal{U}_n$ is isomorphic to the symmetry algebra $\mathcal{U}$ and each h_n is the representation space of $\mathcal{U}$ (index n only shows the place of h in tensor product).
For any $u \in \mathcal{U}$ let's assume the following notation:

$$u_n = I \otimes I \otimes ... \otimes \underbrace{u}_{n-th\ place} \otimes ... \otimes I,$$

where I is the identity operator. Thus, u_n acts non-trivially only in the space h_n.
All other observables are the linear combinations of products of such "trivial" elements.

Our method to solve such quantum models can be realized in two steps:

- To construct the corresponding *L-operator* $L(\lambda) \in \mathcal{M}(k) \otimes \mathcal{U}$ and an auxiliary *R-matrix* $R(\lambda) \in \mathcal{M}(k) \otimes \mathcal{M}(k)$, s.t. the following equation in $\mathcal{M}(k) \otimes \mathcal{M}(k) \otimes \mathcal{U}$ (*Yang-Baxter eq.*)

$$R_{12}(\lambda)L_{13}(\lambda\mu)L_{23}(\mu) = L_{23}(\mu)L_{13}(\lambda\mu)R_{12}(\lambda) \tag{1}$$

  is equivalent to the defining relations of $\mathcal{U}$. $\lambda, \mu \in \mathbb{C}$ are called *spectral parameters*.
  Notation R_{12} means that it acts non-trivially only on the first two components in tensor product. It can be written as $R_{12} = R \otimes I$ (and analogously for L_{13} and L_{23}).
  We get such equation corresponding to each lattice site (they differ only by index n):

$$R_{a_1,a_2}(\lambda)L_{a_1,n}(\lambda\mu)L_{a_2,n}(\mu) = L_{a_2,n}(\mu)L_{a_1,n}(\lambda\mu)R_{a_1,a_2}(\lambda)$$

  (this is the equation in $\mathcal{M}(k)\otimes\mathcal{M}(k)\otimes\mathcal{U}_n$). Indices a_1, a_2 correspond to the auxiliary spaces $\mathcal{M}(k)$, index n correspond to the representation space $\mathcal{U}_n$.
  Let's consider the *monodromy matrix*:

$$T_a(\lambda) = L_{a,N}(\lambda) \; ... \; L_{a,1}(\lambda).$$

  It satisfies the following equation:

$$R_{a_1,a_2}(\lambda)T_{a_1}(\lambda\mu)T_{a_2}(\mu) = T_{a_2}(\mu)T_{a_1}(\lambda\mu)R_{a_1,a_2}(\lambda).$$

  This equation implies that the *trace* of the monodromy matrix is the generating function for mutually commuting IM $\{I_n\}$:

$$\tau(\lambda) = tr_a T_a(\lambda) = \sum \lambda^n I_n.$$

  However, these IM are in general *non-local*, i.e. they involve terms acting non-trivially not only on the several neighbour sites of the lattice. The method for construction of the *local* IM is described in the next step.

- To find the corresponding *fundamental R-operator* $R(\lambda) \in \mathcal{U} \otimes \mathcal{U}$ satisfying the following equation in $\mathcal{M}(k) \otimes \mathcal{U} \otimes \mathcal{U}$:

$$R_{23}(\lambda)L_{13}(\lambda\mu)L_{12}(\mu) = L_{12}(\mu)L_{13}(\lambda\mu)R_{23}(\lambda),$$

  where $L(\lambda)$ is the L-operator constructed in the previous step.
  The trace of the corresponding monodromy matrix

$$\tau(\lambda) = tr_a(R_{a,N}(\lambda) \; ... \; R_{a,1}(\lambda)),$$

  where index a stands for an auxiliary copy of $\mathcal{U}$, generates now the *local* mutually commuting IM (under some regularity conditions).

  In our case it will be more convenient to work with $r(\lambda) \equiv R(\lambda)\mathbb{P}$, where $\mathbb{P}$ is the permutation operator in $\mathcal{U} \otimes \mathcal{U}$: $\mathbb{P}(x \otimes y)\mathbb{P} = y \otimes x$ for any $x, y \in \mathcal{U}$.
  The equation for $r(\lambda)$ takes the following form:

$$r_{23}(\lambda)L_{12}(\lambda\mu)L_{13}(\mu) = L_{12}(\mu)L_{13}(\lambda\mu)r_{23}(\lambda). \tag{2}$$

In this work we'll consider the case of the symmetry algebra $\mathcal{U} = GL_q(2)$, i.e. the q-deformed algebra of functions on the Lie group $GL(2)$. Such objects are usually called *algebras of functions on quantum groups*, or just *quantum groups* (because they don't exist separately from their algebras). They admit the structure of a *bialgebra* (moreover, of a *Hopf algebra*, but we don't need this), which will be very helpful to solve the Eq.(2). More precisely, we'll consider an extended bialgebra $\widetilde{GL}_q(2)$ with one additional generator. The procedure described above was applied to the models with such symmetry algebra for the case $|q| = 1$ (look [1]). So, the main goal of this work is to apply the same procedure to the essentially different case, when $q \in \mathbb{R},\ 0 < q < 1$. In this case there is no *positive* representation of this algebra (where all generators are invertible), that's why some proofs developed for the previous case fail here.

The work is organized as follows. At first I'll give some general information about the quantum groups in general and about $GL_q(2)$ in particular. Then I'll collect some facts about it that we assume to be known and briefly describe the construction of the corresponding L-operator. And, finally, we'll turn to our main problem.

2 Some words about quantum groups

2.1 Algebra of functions on a group

Quantum group can be seen as the q-deformation of an algebra of functions on some group.
So, at first let's consider some group G (usually it's a *Lie group*) and some algebra $\mathcal{A} = \mathcal{F}(G)$ of functions on it. In general, this is a unital associative and *abelian* algebra with multiplication and identity defined by the following formulas:

$$(f_1 f_2)(g) = f_1(g) f_2(g), \quad I(g) \equiv 1.$$

For simplicity, it can be seen as (some extension of) an algebra of polynomials on G.
In this case $\mathcal{F}(G \times G) \simeq \mathcal{A} \otimes \mathcal{A}$ and $\mathcal{A}$ admits the structure of a *Hopf algebra*:

1. *comultiplication* $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}, \quad (\Delta f)(g_1, g_2) = f(g_1 g_2);$
2. *counity* $\epsilon : \mathcal{A} \to \mathbb{C}, \quad \epsilon(f) = f(e);$
3. *antipode* $S : \mathcal{A} \to \mathcal{A}, \quad (Sf)(g) = f(g^{-1}).$

One can check that these operations satisfy all needed conditions.

2.2 Example: $\mathcal{F}(GL(2))$

Let's consider the case $G = GL(2)$, the Lie group of invertible 2×2 - matrices.
Then the corresponding abelian bialgebra $\mathcal{F}(GL(2))$ is generated by so-called *coordinate functions* $\pi_{11},\ \pi_{12},\ \pi_{21},\ \pi_{22}$ acting as follows:

$$\pi_{ij}(g) = g_{ij}, \quad \text{where} \quad g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \in GL(2).$$

Let's assemble these generators into a matrix and choose more convenient notations:

$$\begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix} \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

One can check, that *comultiplication* and *counity* introduced above in this case are:

$$\begin{aligned} \Delta(a) &= a \otimes a + b \otimes c, \quad & \Delta(b) &= a \otimes b + b \otimes d, \\ \Delta(c) &= c \otimes a + d \otimes c, \quad & \Delta(d) &= c \otimes b + d \otimes d, \\ \epsilon(a) &= \epsilon(d) = 1, \quad & \epsilon(b) &= \epsilon(c) = 0. \end{aligned}$$

2.3 q-deformation of $\mathcal{F}(GL(2))$

Definition. $GL_q(2)$ is a unital associative algebra with generators $a,\ b,\ c,\ d$ and defining relations:

$$\begin{aligned} [a,d] = (q-q^{-1})bc, \quad [b,c]=0, \\ ab = qba, \quad ac = qca, \quad bd = qdb, \quad cd = qdc. \end{aligned} \tag{3}$$

Comultiplication and counity are *defined* as in previous section.
Thus, setting $q = 1$ we get the *non-deformed* abelian bialgebra $\mathcal{F}(GL(2))$.

The generators of $GL_q(2)$ can be assembled into a matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
Then the expressions for comultiplication and counity can be rewritten in matrix form:

$$(id \otimes \Delta)g = g_{12}g_{13}, \quad \epsilon(g) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let's consider an element $\mathcal{D}_q = ad - qbc = da - q^{-1}bc$, which is called the *quantum determinant.* One can check that it commutes with all generators, i.e. belongs to the center of $GL_q(2)$. In fact, the center is generated by $\mathcal{D}_q$. Moreover, it is *group-like* with respect to the introduced comultiplication and counity, i.e.

$$\Delta(\mathcal{D}_q) = \mathcal{D}_q \otimes \mathcal{D}_q, \quad \epsilon(\mathcal{D}_q) = 1.$$

2.4 Geometric interpretation of $GL_q(2)$

Lie group $GL(2)$ is a group of invertible transformations of the complex plane $\mathbb{C}^2$. Analogously, the defining relations (3) of $GL_q(2)$ appear, if we interpret the matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as a transformation of the *quantum plane* $\mathbb{C}^2_q$ (i.e. the algebra with generators x, y satisfying $xy = qyx$ $(\star)$).

Let's define the *left* and the *right action* of this matrix:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \otimes x + b \otimes y \\ c \otimes x + d \otimes y \end{pmatrix},$$

$$\begin{pmatrix} x'' & y'' \end{pmatrix} \equiv \begin{pmatrix} x & y \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} x \otimes a + y \otimes c & x \otimes b + y \otimes d \end{pmatrix}.$$

Proposition. Pairs $\begin{pmatrix} x' \\ y' \end{pmatrix}$ and $\begin{pmatrix} x'' & y'' \end{pmatrix}$ satisfy the relation $(\star)$ iff $a,\ b,\ c,\ d$ satisfy (3).

2.5 Definition of $\widetilde{GL}_q(2)$

Definition. $\widetilde{GL}_q(2)$ is a unital associative algebra with generators $a,\ b,\ c,\ d,\ \theta$ satisfying all commutation relations (3) for $GL_q(2)$ and four additional relations for θ :

$$a\theta = q^{-1}\theta a, \quad \theta d = q^{-1}d\theta, \quad [b,\theta] = 0, \quad [\theta, c] = 0. \tag{4}$$

An additional element θ can be chosen as an inverse of b or c. Let's consider the elements:

$$\eta_q^{'} = \theta b, \quad \eta_q^{''} = \theta c.$$

Then the center of $\widetilde{GL}_q(2)$ is generated already by three elements: $\mathcal{D}_q,\ \eta_q^{'},\ \eta_q^{''}$.

3 Preliminaries

In this chapter we'll briefly (without proofs) consider the first step of our procedure, i.e. construction of the corresponding *L-matrices* for $GL_q(2)$ and $\widetilde{GL}_q(2)$. Sometimes it is called *Baxterization* of the algebra. And in the next chapter we'll turn to our main problem of constructing fundamental R-operator.

3.1 Baxterization of $GL_q(2)$

Lemma 1. *The defining relations of $GL_q(2)$ are equivalent to each of the following relations:*

$$R^{\pm}_{12} g_{13} g_{23} = g_{23} g_{13} R^{\pm}_{12},$$

where

$$R^{+} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \quad R^{-} = \begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & 1 & q^{-1}-q & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix}.$$

Let's introduce spectral parameter dependent L-matrices:

$$g(\lambda) = \begin{pmatrix} a & \lambda b \\ \lambda^{-1} c & d \end{pmatrix}, \quad \hat{g}(\lambda) = \begin{pmatrix} \lambda^{-1} c & \lambda^{-1} d \\ \lambda a & \lambda b \end{pmatrix}.$$

Lemma 2. *The defining relations of $GL_q(2)$ are equivalent to each of the following relations:*

$$R_{12}(\lambda) g_{13}(\lambda\mu) g_{23}(\mu) = g_{23}(\mu) g_{13}(\lambda\mu) R_{12}(\lambda),$$
$$\hat{R}_{12}(\lambda) \hat{g}_{13}(\lambda\mu) \hat{g}_{23}(\mu) = \hat{g}_{23}(\mu) \hat{g}_{13}(\lambda\mu) \hat{R}_{12}(\lambda)$$

with

$$\hat{R}(\lambda) = \lambda R^{+} - \lambda^{-1} R^{-} = \begin{pmatrix} \omega(q\lambda) & 0 & 0 & 0 \\ 0 & \omega(\lambda) & \lambda^{-1}\omega(q) & 0 \\ 0 & \lambda\omega(q) & \omega(\lambda) & 0 \\ 0 & 0 & 0 & \omega(q\lambda) \end{pmatrix}, \quad R(\lambda) = \begin{pmatrix} \omega(q\lambda) & 0 & 0 & 0 \\ 0 & \omega(\lambda) & \omega(q) & 0 \\ 0 & \omega(q) & \omega(\lambda) & 0 \\ 0 & 0 & 0 & \omega(q\lambda) \end{pmatrix},$$

where $\omega(\lambda) \equiv \lambda - \lambda^{-1}$.

Rem. The term *Baxterization* was originally introduced exactly for this procedure of constructing spectral parameter dependent L-matrices from the constant solutions of Yang-Baxter equation.

3.2 Baxterization of $\widetilde{GL}_q(2)$

Lemma 1. *The defining relations of $\widetilde{GL}_q(2)$ are equivalent to the following set of relations:*

$$R^{\pm}_{12} g^{\pm}_{13} g^{\pm}_{23} = g^{\pm}_{23} g^{\pm}_{13} R^{\pm}_{12}, \quad R^{+}_{12} g^{+}_{13} g^{-}_{23} = g^{-}_{23} g^{+}_{13} R^{+}_{12},$$

where $g^{+} = \begin{pmatrix} \theta & 0 \\ a & b \end{pmatrix}$ *and* $g^{-} = \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}$, *matrices $R^{\pm}$ are given in previous section.*

On analogy to the previous case let's introduce spectral parameter dependent L-matrices:

$$g(\lambda) = \begin{pmatrix} a & \lambda b \\ \lambda\theta + \lambda^{-1} c & d \end{pmatrix}, \quad \hat{g}(\lambda) = \begin{pmatrix} \lambda\theta + \lambda^{-1} c & \lambda^{-1} d \\ \lambda a & \lambda b \end{pmatrix}.$$

Lemma 2. *The defining relations of $\widetilde{GL}_q(2)$ are equivalent to each of the following relations:*

$$R_{12}(\lambda) g_{13}(\lambda\mu) g_{23}(\mu) = g_{23}(\mu) g_{13}(\lambda\mu) R_{12}(\lambda),$$
$$\hat{R}_{12}(\lambda) \hat{g}_{13}(\lambda\mu) \hat{g}_{23}(\mu) = \hat{g}_{23}(\mu) \hat{g}_{13}(\lambda\mu) \hat{R}_{12}(\lambda),$$

where auxiliary matrices $R(\lambda)$ and $\hat{R}(\lambda)$ are the same as in the previous section.

4 The main problem

As we have already stated, the main problem is to find the fundamental R-operator for already constructed L-operator:

$$\hat{g}(\lambda) = \lambda g^+ + \lambda^{-1} g^- = \begin{pmatrix} \lambda\theta + \lambda^{-1}c & \lambda^{-1}d \\ \lambda a & \lambda b \end{pmatrix} \in \mathcal{M}(2) \otimes \mathcal{U}.$$

The proposition is that, although there is no positive representation for $\mathcal{U}$, the formula from the case $|q| = 1$ (look [1]) remains true. Let's consider the following representation of $GL_q(2)$ ($0 < q < 1$) in some Hilbert space $\mathfrak{H}$ with orthonormal basis $\{e_k\}$, $k = 0, 1, 2, \ldots$:

$$T(a)e_0 = 0, \quad T(a)e_k = \gamma(1 - q^{2k})^{1/2} e_{k-1}, \ k \geq 1,$$
$$T(b)e_k = e^{-i\varphi} q^k e_k, \quad T(c)e_k = -\gamma e^{i\varphi} q^{k+1} e_k, \quad T(\theta)e_k = e^{i\varphi} q^{-k} e_k,$$
$$T(d)e_k = (1 - q^{2k+2})^{1/2} e_{k+1}.$$

Here $0 \leq \varphi < 2\pi$ and $\gamma > 0$ are parameters of the representation. Then we have

$$T(b)T(c)e_k = -\gamma q^{2k+1} e_k, \quad T(b)T(\theta)e_k = e_k, \ \text{i.e. } \theta \text{ is an inverse of } b,$$
$$T(a)T(d)e_k = \gamma(1 - q^{2k+2})e_k, \quad T(d)T(a)e_k = \gamma(1 - q^{2k})e_k,$$
$$T(\mathcal{D}_q)e_k = \gamma e_k.$$

For brevity of notations, we'll write just $x \otimes y$ instead of $T(x) \otimes T(y)$ (when the concrete representation is not very important).

Main Proposition. *The operator $\hat{r}(\lambda) \in \mathcal{U} \otimes \mathcal{U}$ defined by the formula*

$$\hat{r}(\lambda) = (ac \otimes \theta d + bc \otimes D_q)^{\alpha \ln \lambda}, \tag{5}$$

where $\alpha \equiv \frac{1}{\ln q}$, satisfies the equation

$$\hat{r}_{23}(\lambda)\hat{g}_{12}(\lambda\mu)\hat{g}_{13}(\mu) = \hat{g}_{12}(\mu)\hat{g}_{13}(\lambda\mu)\hat{r}_{23}(\lambda). \tag{6}$$

Proof: It remains to prove that $\hat{r}(\lambda)$ satisfies the following equation (look [1]):

$$\hat{r}(\lambda)(\lambda\theta \otimes d + \lambda^{-1} d \otimes b) = (\lambda^{-1}\theta \otimes d + \lambda d \otimes b)\hat{r}(\lambda). \tag{7}$$

At first let's prove this for the integer powers in expression (5) for $\hat{r}(\lambda)$, i.e. for the points $\lambda = q^n$ (then we have $\alpha \ln \lambda = n$). Let's denote

$$u \equiv ac \otimes \theta d, \qquad v \equiv bc \otimes \mathcal{D}_q.$$

One can check that they satisfy the commutation relation $uv = q^2 vu$, i.e. form the *Weyl pair*. Thus, we have

$$\hat{r}(q^n) = (u + v)^n = \sum_{k=0}^{n} \binom{n}{k}_{q^{-2}} u^k v^{n-k},$$

where q^{-2} - *binomial coefficients* are defined by the formula:

$$\binom{n}{k}_{q^{-2}} = \frac{(q^{-2}; q^{-2})_n}{(q^{-2}; q^{-2})_k (q^{-2}; q^{-2})_{n-k}} \tag{8}$$

with

$$(q^{-2}; q^{-2})_k = \prod_{j=1}^{k} (1 - q^{-2j}), \quad (q^{-2}; q^{-2})_0 = 1.$$

We want to prove that

$$(u+v)^n(\lambda\theta\otimes d+\lambda^{-1}d\otimes b)=(\lambda^{-1}\theta\otimes d+\lambda d\otimes b)(u+v)^n,\ \text{where } \lambda=q^n.$$

Let's try to interchange the multiplicands in each term

$$u^k v^{n-k}(\lambda\theta\otimes d+\lambda^{-1}d\otimes b).$$

For v all is rather simple:

$$v(\lambda\theta\otimes d+\lambda^{-1}d\otimes b)=(\lambda\theta\otimes d+\lambda^{-1}q^2 d\otimes b)v,$$

$$v^{n-k}(\lambda\theta\otimes d+\lambda^{-1}d\otimes b)=(\lambda\theta\otimes d+\lambda^{-1}q^{2(n-k)}d\otimes b)v^{n-k}. \qquad (9)$$

For u there are some difficulties, because of the relation $[a,d]=(q-q^{-1})bc$:

$$u(\theta\otimes d)=q^{-2}(\theta\otimes d)u \quad\Rightarrow\quad u^k(\theta\otimes d)=q^{-2k}(\theta\otimes d)u^k,$$

$$u(d\otimes b)=(d\otimes b)u+(q-q^{-1})bc^2\otimes b\theta d.$$

Substituting $bc=q(da-\mathcal{D}_q)$ we get:

$$u(d\otimes b)=q^2(d\otimes b)u-(q^2-1)(\mathcal{D}_q c\otimes b\theta d),$$

$$u^k(d\otimes b)=u^{k-1}(q^2(d\otimes b)u-(q^2-1)(\mathcal{D}_q c\otimes b\theta d))=q^2u^{k-1}(d\otimes b)u-(q^2-1)(\mathcal{D}_q c\otimes b\theta d)u^{k-1}=$$
$$=q^4u^{k-2}(d\otimes b)u^2-q^2(q^2-1)(\mathcal{D}_q c\otimes b\theta d)u^{k-1}-(q^2-1)(\mathcal{D}_q c\otimes b\theta d)u^{k-1}=\ldots$$
$$=q^{2k}(d\otimes b)u^k-(q^{2(k-1)}+\ldots+q^2+1)(q^2-1)(\mathcal{D}_q c\otimes b\theta d)u^{k-1}=q^{2k}(d\otimes b)u^k-(q^{2k}-1)(\mathcal{D}_q c\otimes b\theta d)u^{k-1}.$$

So, we have

$$u^k(\lambda\theta\otimes d+\lambda^{-1}d\otimes b)=(\lambda q^{-2k}\theta\otimes d+\lambda^{-1}q^{2k}d\otimes b)u^k-\lambda^{-1}(q^{2k}-1)(\mathcal{D}_q c\otimes b\theta d)u^{k-1}. \qquad (10)$$

And finally we get:

$$u^kv^{n-k}(\lambda\theta\otimes d+\lambda^{-1}d\otimes b)=u^k(\lambda\theta\otimes d+\lambda^{-1}q^{2(n-k)}d\otimes b)v^{n-k}=$$
$$=(\lambda q^{-2k}\theta\otimes d+\lambda^{-1}q^{2n}d\otimes b)u^kv^{n-k}-\lambda^{-1}q^{2n}(1-q^{-2k})(\mathcal{D}_q c\otimes b\theta d)u^{k-1}v^{n-k}.$$

In particular, for $\lambda=q$ (i.e. we have $\alpha\ln\lambda=1$):

$$(u+v)(q\theta\otimes d+q^{-1}d\otimes b)=(qq^{-2}\theta\otimes d+q^{-1}q^2d\otimes b)u-$$
$$-\ q^{-1}(q^2-1)(\mathcal{D}_q c\otimes b\theta d)+(q\theta\otimes d+q^{-1}q^2d\otimes b)v\ =$$
$$=\ (q^{-1}\theta\otimes d+qd\otimes b)u+(q\theta\otimes d+qd\otimes b)v-(q-q^{-1})\mathcal{D}_q c\otimes b\theta d.$$

We want to prove that

$$(u+v)(q\theta\otimes d+q^{-1}d\otimes b)=(q^{-1}\theta\otimes d+qd\otimes b)(u+v),\ \text{i.e.}$$

$$(q^{-1}\theta\otimes d+qd\otimes b)u+(q\theta\otimes d+qd\otimes b)v-(q-q^{-1})(\mathcal{D}_q c\otimes b\theta d)=(q^{-1}\theta\otimes d+qd\otimes b)(u+v).$$

After opening all brackets and collecting the similar terms we get:

$$(\theta\otimes d)v=\mathcal{D}_q c\otimes b\theta d \quad\Leftrightarrow\quad (\theta b\otimes\mathcal{D}_q)(c\otimes d)=(\mathcal{D}_q\otimes b\theta)(c\otimes d).$$

At this point we have to use our representation:

$$T(\theta b)\otimes T(\mathcal{D}_q)=I\otimes\gamma I=\gamma I\otimes I=T(\mathcal{D}_q)\otimes T(\theta b),$$

So, the action of these two elements is the same and we have proved our formula for the point $\lambda=q$.

Let's now turn to more difficult case $\lambda = q^n$ for $n > 1$. We have already derived the following formula (setting $b\theta = I$):

$$u^k v^{n-k}(q^n \theta \otimes d + q^{-n} d \otimes b) = (q^{n-2k}\theta \otimes d + q^n d \otimes b) u^k v^{n-k} - (q^n - q^{n-2k})(\mathcal{D}_q c \otimes d) u^{k-1} v^{n-k} =$$

$$= (q^{n-2k}\theta \otimes d + q^n d \otimes b) u^k v^{n-k} - (q^n - q^{n-2k}) q^{-2k+2} (\mathcal{D}_q b^{-1} \otimes d \mathcal{D}_q^{-1}) u^{k-1} v^{n-k+1}.$$

Let's substitute it in our sum (taking into account that $b^{-1} = \theta$ and $\mathcal{D}_q = \gamma I$):

$$\sum_{k=0}^{n} \binom{n}{k}_{q^{-2}} u^k v^{n-k} (q^n \theta \otimes d + q^{-n} d \otimes b) = \sum_{k=0}^{n} \binom{n}{k}_{q^{-2}} (q^{n-2k}\theta \otimes d + q^n d \otimes b) u^k v^{n-k} -$$

$$- \sum_{k=0}^{n-1} \binom{n}{k+1}_{q^{-2}} (q^n - q^{n-2k-2}) q^{-2k} (\theta \otimes d) u^k v^{n-k} = q^n (d \otimes b) \sum_{k=0}^{n} \binom{n}{k}_{q^{-2}} u^k v^{n-k} +$$

$$+ q^{-n} (\theta \otimes d) u^n + \sum_{k=0}^{n-1} \left(\binom{n}{k}_{q^{-2}} q^{n-2k} - \binom{n}{k+1}_{q^{-2}} (q^n - q^{n-2k-2}) q^{-2k} \right) (\theta \otimes d) u^k v^{n-k}.$$

We want to prove that it is equal to

$$(q^{-n}\theta \otimes d + q^n d \otimes b) \sum_{k=0}^{n} \binom{n}{k}_{q^{-2}} u^k v^{n-k}.$$

After cancelling the same terms we get that we must prove the following equality:

$$\sum_{k=0}^{n-1} \left(\binom{n}{k}_{q^{-2}} q^{n-2k} - \binom{n}{k+1}_{q^{-2}} (q^n - q^{n-2k-2}) q^{-2k} \right) u^k v^{n-k} = q^{-n} \sum_{k=0}^{n-1} \binom{n}{k}_{q^{-2}} u^k v^{n-k}, \text{ i.e.}$$

$$\binom{n}{k}_{q^{-2}} q^{n-2k} - \binom{n}{k+1}_{q^{-2}} (q^n - q^{n-2k-2}) q^{-2k} = q^{-n} \binom{n}{k}_{q^{-2}} \Leftrightarrow$$

$$\Leftrightarrow \binom{n}{k}_{q^{-2}} (q^{n-2k} - q^{-n}) = \binom{n}{k+1}_{q^{-2}} (q^n - q^{n-2k-2}) q^{-2k} \Leftrightarrow$$

$$\Leftrightarrow \binom{n}{k}_{q^{-2}} (1 - q^{2(k-n)}) = \binom{n}{k+1}_{q^{-2}} (1 - q^{-2(k+1)}).$$

On the other hand, from the formula for q^{-2} - binomial coefficients follows that:

$$\binom{n}{k}_{q^{-2}} (1 - q^{-2(n-k)}) = \binom{n}{k+1}_{q^{-2}} (1 - q^{-2(k+1)}).$$

So, we are done in this case. It remains now to consider the general situation, i.e. an arbitrary λ:

$$\hat{r}(\lambda) = (u+v)^{\alpha \ln \lambda} = \sum_{k=0}^{\infty} \binom{\alpha \ln \lambda}{k}_{q^{-2}} u^k v^{\alpha \ln \lambda - k},$$

where

$$\binom{\alpha \ln \lambda}{k}_{q^{-2}} = \prod_{j=1}^{k} \frac{1 - q^{-2(\alpha \ln \lambda - j + 1)}}{1 - q^{-2j}} = \prod_{j=1}^{k} \frac{1 - \lambda^{-2} q^{2j-2}}{1 - q^{-2j}}.$$

Analogously to the previous case we get the following formula:

$$u^k v^{\alpha \ln \lambda - k}(\lambda \theta \otimes d + \lambda^{-1} d \otimes b) = u^k (\lambda \theta \otimes d + \lambda^{-1} q^{2(\alpha \ln \lambda - k)} d \otimes b) v^{\alpha \ln \lambda - k} =$$

$$= (\lambda q^{-2k} \theta \otimes d + \lambda^{-1} q^{2\alpha \ln \lambda} d \otimes b) u^k v^{\alpha \ln \lambda - k} - \lambda^{-1} q^{2\alpha \ln \lambda} (1 - q^{-2k}) q^{-2k+2} (\theta \otimes d) u^{k-1} v^{\alpha \ln \lambda - k + 1}.$$

Taking into account that $q^{\alpha \ln \lambda} = \lambda$ we get

$$u^k v^{\alpha \ln \lambda - k}(\lambda \theta \otimes d + \lambda^{-1} d \otimes b) = (\lambda q^{-2k} \theta \otimes d + \lambda d \otimes b) u^k v^{\alpha \ln \lambda - k} - \lambda(1 - q^{-2k}) q^{-2k+2} (\theta \otimes d) u^{k-1} v^{\alpha \ln \lambda - k + 1}.$$

Now we substitute this in our sum:

$$\sum_{k=0}^{\infty} \binom{\alpha \ln \lambda}{k}_{q^{-2}} u^k v^{\alpha \ln \lambda - k}(\lambda \theta \otimes d + \lambda^{-1} d \otimes b) = \sum_{k=0}^{\infty} \binom{\alpha \ln \lambda}{k}_{q^{-2}} (\lambda q^{-2k} \theta \otimes d + \lambda d \otimes b) u^k v^{\alpha \ln \lambda - k} -$$

$$- \sum_{k=0}^{\infty} \binom{\alpha \ln \lambda}{k+1}_{q^{-2}} \lambda(1 - q^{-2k-2}) q^{-2k} (\theta \otimes d) u^k v^{\alpha \ln \lambda - k} = \lambda(d \otimes b) \sum_{k=0}^{\infty} \binom{\alpha \ln \lambda}{k}_{q^{-2}} u^k v^{\alpha \ln \lambda - k} +$$

$$+ \lambda(\theta \otimes d) \sum_{k=0}^{\infty} \left(\binom{\alpha \ln \lambda}{k}_{q^{-2}} q^{-2k} - \binom{\alpha \ln \lambda}{k+1}_{q^{-2}} (1 - q^{-2k-2}) q^{-2k} \right) u^k v^{\alpha \ln \lambda - k}.$$

It must be equal to

$$(\lambda^{-1} \theta \otimes d + \lambda d \otimes b) \sum_{k=0}^{\infty} \binom{\alpha \ln \lambda}{k}_{q^{-2}} u^k v^{\alpha \ln \lambda - k}.$$

So, we need to prove the following equality:

$$\lambda \sum_{k=0}^{\infty} \left(\binom{\alpha \ln \lambda}{k}_{q^{-2}} q^{-2k} - \binom{\alpha \ln \lambda}{k+1}_{q^{-2}} (1 - q^{-2k-2}) q^{-2k} \right) u^k v^{\alpha \ln \lambda - k} = \lambda^{-1} \sum_{k=0}^{\infty} \binom{\alpha \ln \lambda}{k}_{q^{-2}} u^k v^{\alpha \ln \lambda - k}, \text{ i.e.}$$

$$\binom{\alpha \ln \lambda}{k}_{q^{-2}} q^{-2k} - \binom{\alpha \ln \lambda}{k+1}_{q^{-2}} (1 - q^{-2k-2}) q^{-2k} = \lambda^{-2} \binom{\alpha \ln \lambda}{k}_{q^{-2}} \Leftrightarrow$$

$$\Leftrightarrow \binom{\alpha \ln \lambda}{k}_{q^{-2}} (q^{-2k} - \lambda^{-2}) = \binom{\alpha \ln \lambda}{k+1}_{q^{-2}} (1 - q^{-2k-2}) q^{-2k} \Leftrightarrow$$

$$\Leftrightarrow \binom{\alpha \ln \lambda}{k}_{q^{-2}} (1 - \lambda^{-2} q^{2k}) = \binom{\alpha \ln \lambda}{k+1}_{q^{-2}} (1 - q^{-2k-2}).$$

One can easily check that this equality follows from the definition of q^{-2} - binomial coefficients in this case. So, our main proposition is proved.

5 Appendix: q-oscillator

5.1 Non-deformed case

The algebra of the harmonical oscillator is generated by 2 elements a^+ and a satisfying following commutation relation:

$$[a, a^+] = 1, \text{ i.e. } aa^+ = a^+ a + 1.$$

Using this relation we get:

$$[a^n, a^+] = a^n a^+ - a^+ a^n = a^{n-1}(a^+ a + 1) - a^+ a^n = a^{n-1} a^+ a + a^{n-1} - a^+ a^n = ...$$

$$... = a^+ a^n + n a^{n-1} - a^+ a^n = n a^{n-1}.$$

In general:

$$[f(a), a^+] = f^{'}(a).$$

5.2 q-deformed case

The q-oscillator algebra is generated by elements $a^+,\ a,\ q^N$ with folowing relations:

$$[a,a^+]=q^N,\quad q^Na^+=qa^+q^N,\quad q^Na=q^{-1}aq^N.$$

Under this assumptions one can derive the following formula:

$$[a^n,a^+]=a^na^+-a^+a^n=a^{n-1}(a^+a+q^N)-a^+a^n=a^{n-1}a^+a+a^{n-1}q^N-a^+a^n=...$$

$$...=a^+a^n+a^{n-1}q^N+a^{n-2}q^Na+...+aq^Na^{n-2}+q^Na^{n-1}-a^+a^n=$$

$$=[q^Na^k=q^{-k}a^kq^N]=(q^{n-1}+...+q^2+q+1)q^Na^{n-1}=\frac{1-q^n}{1-q}q^Na^{n-1},\ \text{if}\ \ q\neq 1.$$

6 Conclusion

7 References

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